Math 1100 — Calculus, Quiz #8A — 2009-11-30

(50) 1. Compute $\lim_{x\to\infty} x^4 \cdot \exp(-x^2)$.

Solution: Let $f(x) := x^4$ and $g(x) := \exp(-x^2)$. Observe that $\lim_{x\to\infty} x^4 = \infty$ and $\lim_{x\to\infty} \exp(-x^2) = \lim_{y\to-\infty} \exp(y) = 0$ (making the change of variables $y = -x^2$). Thus, the limit is an indeterminate product of type $\infty \cdot 0$. Let $G(x) := \exp(x^2)$; then $g(x) = 1/G(x)$, and $\lim_{x\to\infty} G(x) = 0$. We have:

$$\begin{aligned}
\lim_{x\to\infty} x^4 \cdot \exp(-x^2) &= \overbrace{\lim_{x\to\infty} \frac{x^4}{\exp(x^2)}}^{\text{(Indet. form of type } \infty/\infty)} \underset{\text{(H1)}}{=} \lim_{x\to\infty} \frac{4x^3}{2x \cdot \exp(x^2)} = \overbrace{\lim_{x\to\infty} \frac{2x^2}{\exp(x^2)}}^{\text{(Indet. form of type } \infty/\infty)} \\
&\underset{\text{(H2)}}{=} \lim_{x\to\infty} \frac{4x}{2x \cdot \exp(x^2)} = \lim_{x\to\infty} \frac{2}{\exp(x^2)} \underset{(*)}{=} \lim_{y\to\infty} \frac{2}{e^y} = \lim_{y\to\infty} 2e^{-y} \\
&= \boxed{0.}
\end{aligned}$$

Here, (H1) is l'Hospital's rule, with $f(x) := x^4$ so that $f'(x) = 4x^3$, while $G(x) := \exp(x^2)$ so that $G'(x) = 2x\exp(x^2)$. Next (H2) is l'Hospital's rule, with $f_1(x) := 2x^2$ so that $f_1'(x) = 4x$, while $G(x) := \exp(x^2)$ so that again $G'(x) = 2x\exp(x^2)$. Finally, $(*)$ is the change of variables $y = x^2$, so that $y\to\infty$ as $x\to\infty$. □

(50) 2. Compute $\lim_{x\to\infty} x^{1/x}$.

Solution: We have $\lim_{x\to\infty} x = \infty$, whereas $\lim_{x\to\infty} \frac{1}{x} = 0$. Thus, this limit is an indeterminate form of type ∞^0. We take the logarithm and apply l'Hospital's rule. We have:

$$\begin{aligned}
\ln\left(x^{1/x}\right) &= \frac{1}{x}\ln(x). \qquad \text{Thus,} \\
\ln\left(\lim_{x\to\infty} x^{1/x}\right) &= \lim_{x\to\infty} \ln\left(x^{1/x}\right) = \lim_{x\to\infty} \frac{\ln(x)}{x} \qquad \text{(indet. form of type } \tfrac{\infty}{\infty}) \\
&\underset{\text{(H)}}{=} \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.
\end{aligned}$$

Here, (H) is l'Hospital's rule, with $f(x) = \ln(x)$ so that $f'(x) = 1/x$, and $g(x) = x$ so that $g'(x) = 1$.

This, of course, is the limit of the *logarithm* of the original expression. It follows that $\lim_{x\to\infty} x^{1/x} = \exp(0) = \boxed{1.}$ □