Math 1100 — Calculus, Quiz #7A — 2009-11-23

(50) 1. Let $f(x) := \frac{1}{5}x^5 + x + 5$. Show that f has exactly one real root (i.e. there is exactly one point $r \in \mathbb{R}$ such that $f(r) = 0$).

Solution: *Existence.* Note that $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. Thus, there exists some (large enough) $x \in \mathbb{R}$ such that $f(-x) < 0 < f(x)$. Thus, since f is continuous, the Intermediate Value Theorem says there is at least one point $r \in [-x, x]$ such that $f(r) = 0$.

Uniqueness. (by contradiction) Suppose there were two points $r_1 < r_2$ in $\mathbb{R}$ such that $f(r_1) = 0 = f(r_2)$. Then Rolle's Theorem says there is some $x \in [r_1, r_2]$ such that $f'(x) = 0$. But $f'(x) = x^4+1$. Thus, $f'(x) = 0$ if and only if $x^4 = -1$, and there is no $x \in \mathbb{R}$ such that $x^4 = -1$. Thus, f *cannot* have two distinct roots —there can be only one. □

(50) 2. Define $f : [0,3] \longrightarrow \mathbb{R}$ by $f(x) := 3x^2 - 12x + 5$. Find the global maximum and global minimum of f on the interval $[0, 3]$.

Solution: If $f(x) := 3x^2 - 12x + 5$, then $f'(x) = 6x - 12$, for all $x \in (0, 3)$. Thus, f is differentiable everywhere on $(0, 3)$. Thus the only critical point of f is the root of f' —namely $x = 2$. To identify the global maximum and global minimum of f we compute the value of f at the critical point 1, and also at both endpoints. We have:

$$\begin{aligned} f(2) &= 12 - 24 + 5 &= -7; \\ f(3) &= 27 - 36 + 5 &= -4; \\ f(0) &= 0 + 0 + 5 &= 5. \end{aligned}$$

Thus, the $\boxed{\text{global maximum is at the endpoint } 0,}$ and the $\boxed{\text{global minimum is at the critical point } 2.}$
□