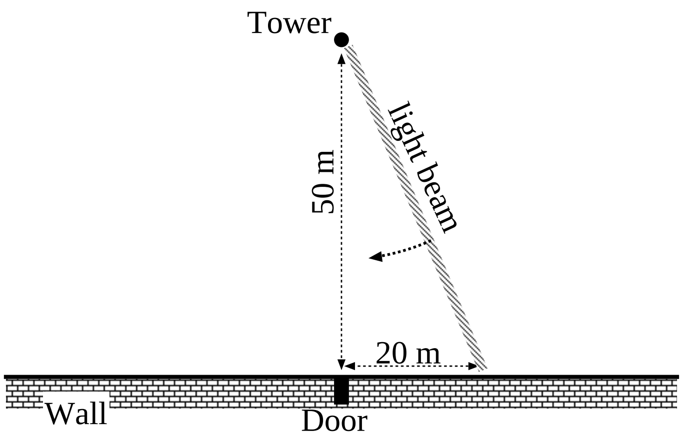
Math 1100 — Calculus, Quiz #6B — 2009-11-19



A searchlight on top of a tower rotates at a constant speed of 1 revolution per minute. There is a brick wall 50 m from the tower, and there is a road stretching in a straight line from the tower to the closest point on the wall, where there is a door. The searchlight makes a spot of light on the wall. How fast is this spot of light moving towards the door when it is 20 m from the door?

(**Hint:** $\arctan'(x) = \frac{1}{1+x^2}$)

Solution: Let $x(t) :=$ the distance from the light spot to the door. We want to solve for $x'(t)$ when $x(t) = 20$.

Let $\theta(t) :=$ the angle that the searchlight beam makes with road. Then $\tan(\theta(t)) = x(t)/50$. Thus, $\theta(t) = \arctan(x(t)/50)$. Differentiating both sides, we have:

$$\theta'(t) \quad = \quad \arctan'\left(\frac{x(t)}{50}\right) \cdot \frac{x'(t)}{50} \quad = \quad \frac{\frac{x'(t)}{50}}{1 + \left(\frac{x(t)}{50}\right)^2} \quad = \quad \frac{50\,x'(t)}{2500 + x(t)^2}.$$

We know that $\theta'(t) = 2\pi/60$ radians/second, because the light beam completes one revolution (i.e. 2π radians) per minute (i.e. 60 seconds). We also know that $x(t) = 20$. Substituting all this, we have:

$$\frac{2\pi}{60} \quad = \quad \frac{50\,x'(t)}{2500 + 20^2} \quad = \quad \frac{50\,x'(t)}{2500 + 400} \quad = \quad \frac{50\,x'(t)}{2900}.$$

Simplifying, we get $x'(t) \;=\; \frac{2 \cdot 2900 \cdot \pi}{60 \cdot 50} \;=\; \boxed{\frac{29\pi}{15}}$ m/s. □