Math 1100 — Calculus, Quiz #18B — 2010-04-8

Are the following series absolutely convergent, conditionally convergent, or divergent? Justify your answer in each case.

(25) 1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\,(\ln(n))^2}$.

Solution: This series is $\boxed{\text{absolutely convergent.}}$ The Integral Test says that the series converges if and only if the improper integral $\displaystyle\int_2^{\infty} \frac{1}{x(\ln(x))^2}\,\mathrm{d}x$ converges. But

$$\int_2^{\infty} \frac{1}{x(\ln(x))^2}\,\mathrm{d}x \underset{(*)}{=} \int_{\ln(2)}^{\infty} \frac{1}{u^2}\,\mathrm{d}u = \lim_{N\to\infty} \frac{-1}{u}\Big|_{u=\ln(2)}^{u=N}$$
$$= \lim_{N\to\infty} \left(\frac{1}{\ln(2)} - \frac{1}{N}\right) = \frac{1}{\ln(2)} < \infty.$$

Thus, the integral is convergent, and thus, so is the series. Here $(*)$ is the change of variables $u := \ln(x)$ so that $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$. □

(25) 2. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$.

Solution: This series is $\boxed{\text{absolutely convergent}}$. To see this, we use the Ratio Test. Let $a_n := \dfrac{n!}{e^{n^2}}$. Then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!/e^{(n+1)^2}}{n!/e^{n^2}} = \frac{(n+1)!}{n!} \cdot \frac{e^{n^2}}{e^{(n^2+2n+1)}}$$
$$= (n+1)\cdot e^{n^2-(n^2+2n+1)} = (n+1)\cdot e^{-2n-1}.$$

Thus, $$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)}{e^{2n+1}} \underset{(*)}{=} \lim_{n\to\infty} \frac{1}{2\,e^{2n+1}} = 0 < 1,$$

where $(*)$ is by l'Hospital's rule. Thus, the Ratio Test says the series converges absolutely. □

(25) 3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n)}$.

Solution: This series is $\boxed{\text{conditionally convergent}}$ but *not* absolutely convergent. To see this, first observe that the sequence $\left\{\dfrac{1}{\ln(n)}\right\}_{n=1}^{\infty}$ is decreasing (because the function $\ln(x)$ is increasing). Also

$$\lim_{n\to\infty} \frac{1}{\ln(n)} = 0.$$

Thus, the Alternating Series Test says that the series converges. However, the series does *not* converge absolutely. To see this, we use the Comparison Test to compare the series $\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ to the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$. For all $n \geq 2$, we have $\ln(n) < n$; thus, $\frac{1}{\ln(n)} > \frac{1}{n}$. Thus, as $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we conclude that $\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ also diverges. □

(25) 4. What is the radius of convergence of the power series $f(x) := \sum_{n=0}^{\infty} \frac{n^2 x^n}{3^n}$?

(*Hint.* Use the Ratio Test to solve for the largest $|x|$ such that the series is absolutely convergent.)

Solution: The radius of convergence is $\boxed{R = 3}$. To see this, we use the Ratio Test. Let $x \in \mathbb{R}$ and define $a_n := \frac{n^2 x^n}{3^n}$. Then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2 x^{n+1}/3^{n+1}}{n^2 x^n/3^n} = \frac{(n+1)^2 x}{n^2} \cdot \frac{3^n}{3^{n+1}} = \frac{(n+1)^2 x}{3n^2}.$$

Thus, $$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^2 x}{3n^2}. = \frac{x}{3} \cdot \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = \frac{x}{3}.$$

Thus,

$$\Big(|x| < 3\Big) \iff \left(\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1\right) \Longrightarrow \Big(\text{Series converges absolutely}\Big), \quad \text{and}$$
$$\Big(|x| > 3\Big) \iff \left(\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1\right) \Longrightarrow \Big(\text{Series diverges}\Big).$$

Thus, the radius of convergence is $R = 3$. □