Math 1100 — Calculus, Quiz #17A — 2010-03-29

1. Consider the parametric curve parameterized by the functions $x(t) := 1+3t^2$ and $y(t) := 4+2t^3$, for all $t \in [-1,1]$.

(10) (a) Compute $x'(t)$ and $y'(t)$.

Solution: $x'(t) = 6t$ and $y'(t) = 6t^2$. $\square$

(10) (b) Find an expression for the *slope* of the curve at the point $(x(t), y(t))$, as a function of t.

Solution: $\text{slope}(t) = \frac{y'(t)}{x'(t)} = \frac{6t^2}{6t} = t.$ $\square$

(10) (c) For what value(s) of t is the tangent line of the curve *horizontal*? For what value(s) of t is the tangent line of the curve *vertical*?

Solution: The tangent is horizontal when $\text{slope}(t)$ is zero, which occurs if and only if $t = 0$.

The tangent is vertical when $\text{slope}(t)$ is infinite, which never occurs for any value of t. $\square$

(25) (d) Compute the *arc length* of the curve between $t = 0$ and $t = 1$.

Solution:

$$\begin{aligned}
\text{arclength} &= \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\ \mathrm{d}t \;=\; \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\ \mathrm{d}t \;=\; \int_0^1 \sqrt{36t^2 + 36t^4}\ \mathrm{d}t \\
&= \int_0^1 6t\sqrt{1+t^2}\ \mathrm{d}t \;\underset{(*)}{=}\; \int_1^2 3\sqrt{u}\ \mathrm{d}u \;=\; 3\frac{2}{3}u^{3/2}\Big|_{u=1}^{u=2} \\
&= 2\cdot\left(2^{3/2} - 1\right) \;=\; \boxed{2^{5/2} - 2.}
\end{aligned}$$

Here, $(*)$ is the change of variables $u = 1 + t^2$, so that $\mathrm{d}u = 2t\ \mathrm{d}t$. $\square$

2. Let $a_n := \frac{n+1}{3n}$ for all $n \in \mathbb{N}$. (So $a_1 = \frac{2}{3},\ a_2 = \frac{1}{2},\ a_3 = \frac{4}{9}$, etc.)

(10) (a) Is the sequence $\{a_n\}_{n=1}^{\infty}$ convergent? If so, what is its limit?

Solution: Yes, the sequence is convergent. We have:

$$\lim_{n\to\infty} \frac{n+1}{3n} = \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{3n}\right) = \frac{1}{3} + \frac{1}{3}\lim_{n\to\infty}\frac{1}{n} = \frac{1}{3} + 0 = \boxed{\frac{1}{3}.}$$

$\square$

(10) (b) Is the series $\sum_{n=1}^{\infty} a_n$ convergent or divergent? If it is convergent, what is its sum?

Solution: The series is $\boxed{\text{divergent,}}$ because $\lim_{n\to\infty} a_n \neq 0$. $\square$

(25) 3. Is the series $\displaystyle\sum_{n=0}^{\infty} \frac{1+2^n}{3^n}$ convergent or divergent? If it is convergent, what is its limit?

Solution: This series is $\boxed{\text{convergent.}}$ We have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1+2^n}{3^n} &= \sum_{n=0}^{\infty} \left(\frac{1}{3^n} + \frac{2^n}{3^n}\right) = \sum_{n=0}^{\infty} \frac{1}{3^n} + \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \\ &= \frac{1}{1-\frac{1}{3}} + \frac{1}{1-\frac{2}{3}} = \frac{1}{2/3} + \frac{1}{1/3} = \frac{3}{2} + 3 = \boxed{\frac{9}{2}.}\end{aligned}$$

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