Math 1100 — Calculus, Quiz #15A — 2010-03-08

Determine whether each of the following improper integrals is convergent or divergent. If it is convergent, then compute its exact value.

(50) 1. $\displaystyle\int_1^\infty \frac{1+\sin(x)^2}{\sqrt{x}}\,dx$.

Solution: This integral is $\boxed{\text{divergent.}}$ To see this, observe that $1+\sin(x)^2 \geq 1$ for all $x \in \mathbb{R}$. Thus,

$$\frac{1+\sin(x)^2}{\sqrt{x}} \quad\geq\quad \frac{1}{\sqrt{x}} \quad=\quad \frac{1}{x^{1/2}}$$

for all $x \in \mathbb{R}$. However, $\displaystyle\int_1^\infty \frac{1}{x^{1/2}}\,dx = \infty$ (because $1/2 \leq 1$). Thus, $\displaystyle\int_1^\infty \frac{1+\sin(x)^2}{\sqrt{x}}\,dx = \infty$, by the Comparison Test. □

(50) 2. $\displaystyle\int_0^\infty x\,\exp(-x^2)\,dx$.

Solution: Let $u := x^2$; then $du = 2x\,dx$. Thus,

$$\int_0^\infty x\,\exp(-x^2)\,dx \quad=\quad \frac{1}{2}\int_0^\infty \exp(-u)\,dx \quad=\quad \frac{1}{2}\lim_{b\to\infty}\int_0^b \exp(-u)\,dx$$

$$= \lim_{b\to\infty} \frac{-e^{-u}}{2}\Big|_{u=0}^{u=b} \quad=\quad \lim_{b\to\infty}\frac{-e^{-b}+1}{2} \quad=\quad \boxed{\frac{1}{2}.}$$

□