Math 1100 — Calculus, Quiz #14B — 2010-03-04

(50) 1. Compute $\displaystyle\int \frac{x-18}{(x+4)(x-3)}\,dx$.

Solution: We wish to find constants A and B such that

$$\frac{x-18}{(x+4)(x-3)} = \frac{A}{x+4}+\frac{B}{x-3} = \frac{A(x-3)+B(x+4)}{(x+4)(x-3)} = \frac{(A+B)\,x+(4B-3A)}{(x+4)(x-2)}.$$

Thus, we need $(A+B)x+(4B-3A) = x+18$, which is equivalent to the system of linear equations

$$A+B = 1; \quad (1)$$
$$4B-3A = -18. \quad (2)$$

Adding 3 times equation (1) to equation (2), we get

$$7B+0A = -15.$$

Thus, $B = -15/7$. Substituting this into equation (1), we get $A - 15/7 = 1$; hence $A = 22/7$. Putting it together, we have

$$\frac{x-18}{(x+4)(x-3)} = \frac{22/7}{x+4} - \frac{15/7}{x-3}.$$

Thus, $$\int \frac{x+18}{(x+4)(x-3)}\,dx = \int \frac{22/7}{x+4} - \frac{15/7}{x-3}\,dx = \int \frac{22/7}{x+4}\,dx - \int \frac{15/7}{x-3}\,dx$$
$$= \boxed{\frac{22}{7}\ln|x+4| - \frac{15}{7}\ln|x-3| + C.}$$

□

(50) 2. Compute $\displaystyle\int \frac{x^3}{\sqrt{36-x^2}}\,dx$. (**Hint:** $3\times 6^3 = 648$.)

Solution: We make the trigonometric substitution $x := 6\sin(\theta)$, so that $dx = 6\cos(\theta)\,d\theta$. Meanwhile,

$$36-x^2 = 36-36\,\sin(\theta)^2 = 36\,\left(1-\sin(\theta)^2\right) = 36\,\cos(\theta)^2.$$

Thus, $$\sqrt{36-x^2} = \sqrt{36\,\cos(\theta)^2} = 6\,|\cos(\theta)| = 6\,\cos(\theta),$$

where the last step is true as long as $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (so that $\cos(\theta) > 0$).

Substituting all this into the integral, we get:

$$\int \frac{x^3}{\sqrt{36-x^2}}\,dx = \int \frac{6^3\sin(\theta)^3}{6\,\cos(\theta)}\cdot 6\cos(\theta)\,d\theta = 6^3\int \sin(\theta)^3\,d\theta$$
$$= 6^3\int \sin(\theta)^2\cdot\sin(\theta)\,d\theta = 6^3\int \left(1-\cos(\theta)^2\right)\cdot\sin(\theta)\,d\theta$$
$$\underset{(*)}{=} 6^3\int (1-u^2)\,du = 6^3\left(u-\frac{u^3}{3}\right) + C \underset{(*)}{=} 6^3\left(\cos(\theta)-\frac{\cos(\theta)^3}{3}\right) + C$$
$$\underset{(\dagger)}{=} \boxed{36\sqrt{36-x^2} - \frac{(36-x^2)^{3/2}}{3} + C.}$$

$(*)$ is the substitution $u := 6\sin(\theta)$ so that $du = 6\cos(\theta)\ d\theta$. Finally, in step $(\dagger)$, we observe that, if $x = 6\sin(\theta)$, then $\sin(\theta) = x/6$, so that $\cos(\theta) = \frac{1}{6}\sqrt{36 - x^2}$ (draw a triangle to see this). $\square$