Math 1100 — Calculus, Quiz #14A — 2010-03-01

(50) 1. Compute $\displaystyle\int \frac{x-7}{(x+5)(x-2)}\,dx$.

Solution: We wish to find constants A and B such that

$$\frac{x-7}{(x+5)(x-2)} = \frac{A}{x+5}+\frac{B}{x-2} = \frac{A(x-2)+B(x+5)}{(x+5)(x-2)} = \frac{(A+B)\,x+(5B-2A)}{(x+5)(x-2)}.$$

Thus, we need $(A+B)x+(5B-2A) = x+12$, which is equivalent to the system of linear equations

$$A+B = 1; \quad (1)$$

$$5B-2A = -7. \quad (2)$$

Adding 2 times equation (1) to equation (2), we get

$$7B+0A = -5.$$

Thus, $B=-5/7$. Substituting this into equation (1), we get $A-5/7=1$; hence $A=12/7$. Putting it together, we have

$$\frac{x-7}{(x+5)(x-2)} = \frac{12/7}{x+5}-\frac{5/7}{x-2}.$$

Thus, $$\int \frac{x+12}{(x+5)(x-2)}\,dx = \int \frac{12/7}{x+5}-\frac{5/7}{x-2}\,dx = \int \frac{12/7}{x+5}\,dx - \int \frac{5/7}{x-2}\,dx$$

$$= \boxed{\frac{12}{7}\ln|x+5| - \frac{5}{7}\ln|x-2| + C.}$$

□

(50) 2. Compute $\displaystyle\int \frac{x^3}{\sqrt{x^2+25}}\,dx$. (**Hint:** $3\times 5^3 = 375$.)

Solution: We make the trigonometric substitution $x := 5\tan(\theta)$, so that $dx = 5\sec(\theta)^2\,d\theta$. Meanwhile,

$$x^2+25 = 25\tan(\theta)^2+25 = 25\left(\tan(\theta)^2+1\right) = 25\sec(\theta)^2.$$

Thus, $$\sqrt{x^2+25} = \sqrt{25\sec(\theta)^2} = 5\,|\sec(\theta)| = 5\sec(\theta),$$

where the last step is true as long as $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ (so that $\sec(\theta)=1/\cos(\theta)>0$.)

Substituting all this into the integral, we get:

$$\int \frac{x^3}{\sqrt{x^2+25}}\,dx = \int \frac{125\tan(\theta)^3}{5\sec(\theta)}\cdot 5\sec(\theta)^2\,d\theta = 125\int \tan(\theta)^3\cdot\sec(\theta)\,d\theta$$

$$= 125\int \tan(\theta)^2\cdot\tan(\theta)\sec(\theta)\,d\theta = 125\int (\sec(\theta)^2-1)\cdot\tan(\theta)\sec(\theta)\,d\theta$$

$$\underset{(*)}{=} 125\int (u^2-1)\,du = 125\left(\frac{u^3}{3}-u\right)+C \underset{(*)}{=} 125\left(\frac{\sec(\theta)^3}{3}-\sec(\theta)\right)+C$$

$$\underset{(\dagger)}{=} \boxed{\frac{(x^2+25)^{3/2}}{3} - 25\sqrt{x^2+25} + C.}$$

$(*)$ is the substitution $u := \sec(\theta)$ so that $du = \sec(\theta)\tan(\theta)\ d\theta$. Finally, in step $(\dagger)$, we observe that, if $x = 5\tan(\theta)$, then $\tan(\theta) = x/5$, so that $\sec(\theta) = 1/\cos(\theta) = \frac{1}{5}\sqrt{x^2+25}$ (draw a triangle to see this). $\square$