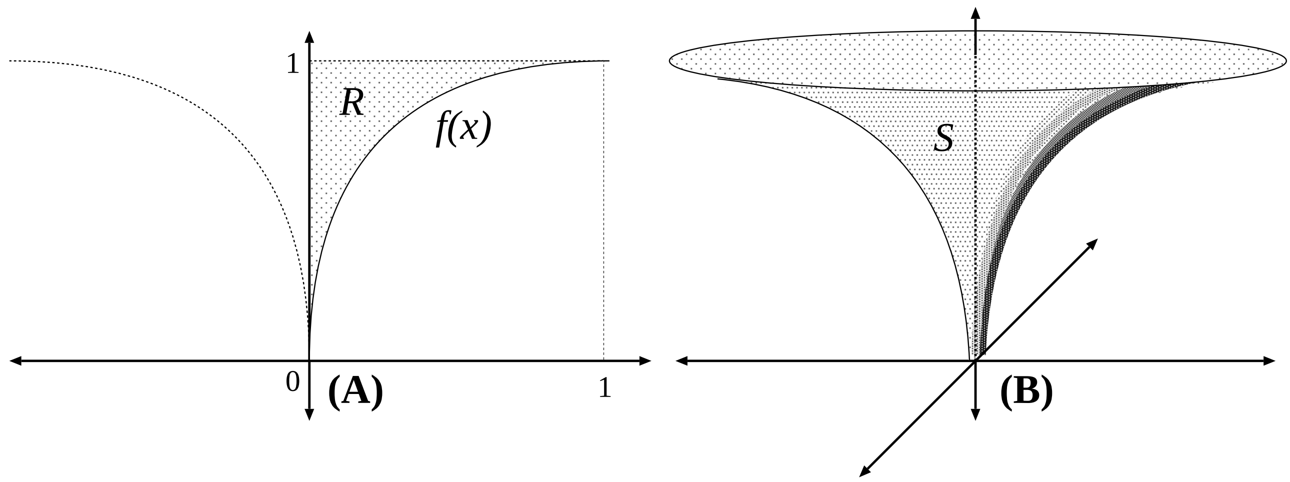
Math 1100 — Calculus, Quiz #13B — 2010-02-12



Let $f(x) := \sqrt[3]{x}$ for all $x \geq 0$, and consider the two-dimensional region $\mathcal{R}$ defined by the constraints $f(x) \leq y \leq 1$ and $0 \leq x \leq 1$ (Figure A). Let $\mathcal{S}$ be the 3-dimensional solid obtained by rotating the region $\mathcal{R}$ around the y axis (Figure B).

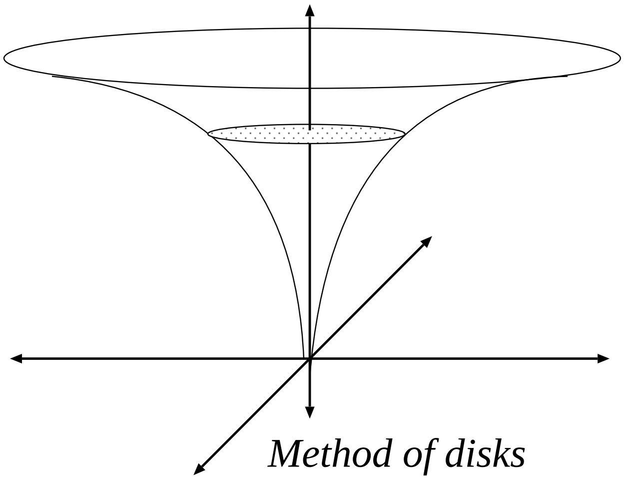
(50) 1. Compute the volume of $\mathcal{S}$ using the *method of disks*. Draw a picture of the typical 'disk' cross-section of $\mathcal{S}$.

Solution: The bounds $0 \leq x \leq 1$ translate into bounds $0 \leq y \leq 1$. To apply method of disks, we must express x as a function of y. If $y = f(x) = \sqrt[3]{x}$, then $x = f^{-1}(y) = y^3$. The area of the disk at height y is $\pi\,(f^{-1}(y))^2 = \pi\,(y^3)^2 \;=\; \pi\,y^6$. We will integrate the areas of these disks as y ranges from 0 to 1. Thus, we have

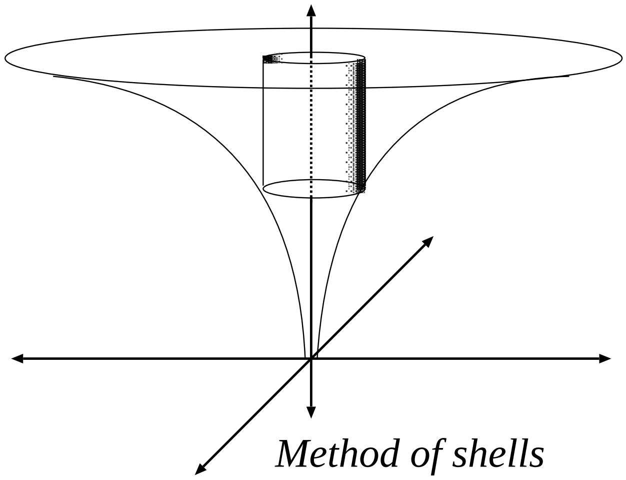
$$V \;=\; \pi \int_0^1 y^6\,dy \;=\; \frac{\pi}{7} y^7 \Big|_{y=0}^{y=1} \;=\; \boxed{\frac{\pi}{7}.}$$

□

2. Compute the volume of $\mathcal{S}$ again, this time using the *method of cylindrical shells*. Draw
(50) a picture of the typical 'cylindrical shell' in $\mathcal{S}$. (*Caution:* $\mathcal{R}$ is the area *above* the curve $y = f(x)$, not *below* this curve.)



Method of disks



Method of shells

Solution: For all $x \in [0,1]$, the cylinder of radius x is generated by the vertical line segment $f(x) \leq y \leq 1$, which has height $(1 - f(x))$, and hence, surface area $2\pi x\,(1 - f(x)) = 2\pi x(1 - \sqrt[3]{x}) = 2\pi(x - x^{4/3})$. We must integrate these areas from $x = 0$ to $x = 1$. Thus,

$$\begin{aligned} V &= 2\pi \int_0^1 (x - x^{4/3})\ dx \quad = \quad 2\pi \left(\frac{x^2}{2} - \frac{3\,x^{7/3}}{7} \right)_{x=0}^{x=1} \\ &= 2\pi \left(\frac{1}{2} - \frac{3}{7} \right) \quad = \quad \pi - \frac{6\pi}{7} \quad = \quad \boxed{\frac{\pi}{7},} \end{aligned}$$

in agreement with the answer to question #1. $\square$