Math 1100 — Calculus, Quiz #11B — 2010-01-28

Consider the function $f(x) = x^{1/3} - \sin(x)$.

(20) 1. Define $F(y) := \int_0^y x^{1/3} - \sin(x)\, dx$ for all $y \in \mathbb{R}$. Compute $F'(y)$.

Solution: The Fundamental Theorem of Calculus says $\boxed{F'(y) = f(y) = y^{1/3} - \sin(y)}$ for all $y \in \mathbb{R}$. □

(20) 2. Define $G(y) := \int_0^{\ln(y)} x^{1/3} - \sin(x)\, dx$ for all $y \in \mathbb{R}$. Compute $G'(y)$.

Solution: Observe that $G(y) = F(\ln(y))$. Thus,

$$G'(y) \underset{(c)}{=} F'(\ln(y)) \cdot \ln'(y) \underset{(*)}{=} \boxed{\left(\ln(y)^{1/3} - \sin[\ln(y)]\right) \cdot \frac{1}{y},}$$

where (c) is the Chain Rule, and $(*)$ is by question #1. □

(20) 3. Express the integral $\int_0^2 x^{1/3} - \sin(x)\, dx$ as a limit of Riemann sums (do not evaluate this limit).

Solution: By definition, $\int_a^b f(x) := \lim_{N\to\infty} \sum_{n=1}^N f(x_{N;n})\Delta_N$, where $\Delta_N := \frac{b-a}{N}$ and where $x_{N;n} := a + n\,\Delta_N$ for all $n \in [1 \ldots N]$. In this case, $a = 0$ and $b = 2$, so $\Delta_N = 2/N$ and $x_{N;n} = 2n/N$. Thus, we have

$$\int_0^2 x^{1/3} - \sin(x)\, dx = \lim_{N\to\infty} \sum_{n=1}^N \left(x_{N;n}^{1/3} - \sin(x_{N;n})\right) \cdot \Delta_N = \boxed{\lim_{N\to\infty} \frac{2}{N} \sum_{n=1}^N \left[\left(\frac{2n}{N}\right)^{1/3} - \sin\left(\frac{2n}{N}\right)\right]}.$$

(It would also be correct to sum from $n = 0$ to $n = N - 1$. More generally, it would be correct to define the integral using any 'sample points' $x^*_{N,1}, x^*_{N,2}, \ldots, x^*_{N,N}$ such that $x^*_{N,n} \in [x_{N,n-1}, x_{N,n}]$ for all $n \in [1...N]$). □

(20) 4. Compute the general antiderivative of $f(x)$.

Solution: The general antiderivative is the function $F(x) = \boxed{\frac{3}{4}x^{4/3} + \cos(x) + C.}$ □

(20) 5. Compute the value of $\int_0^2 x^{1/3} - \sin(x)\, dx$.

Solution: The Fundamental Theorem of Calculus says

$$\begin{aligned}\int_0^2 x^{1/3} - \sin(x) &= F(2) - F(0) = \left(\frac{3}{4}2^{4/3} + \cos(2)\right) - \left(\frac{3}{4}0^{4/3} + \cos(0)\right) \\ &= \boxed{\frac{3\sqrt[3]{2}}{2} + \cos(2) - 1.}\end{aligned}$$

□