Math 1100 — Calculus, Quiz #11A — 2010-01-25

Consider the function $f(x) = x^3 + e^x$.

(20) 1. Define $F(y) := \int_0^y x^3 + e^x \, dx$ for all $y \in \mathbb{R}$. Compute $F'(y)$.

Solution: The Fundamental Theorem of Calculus says $\boxed{F'(y) = f(y) = y^3 + e^y}$ for all $y \in \mathbb{R}$. □

(20) 2. Define $G(y) := \int_0^{\tan(y)} x^3 + e^x \, dx$ for all $y \in \mathbb{R}$. Compute $G'(y)$.

Solution: Observe that $G(y) = F(\tan(y))$. Thus,

$$G'(y) \underset{(c)}{=} F'(\tan(y)) \cdot \tan'(y) \underset{(*)}{=} \boxed{\left(\tan(y)^3 + e^{\tan(y)}\right) \cdot \sec^2(y),}$$

where (c) is the Chain Rule, and $(*)$ is by question #1 □

(20) 3. Express the integral $\int_0^2 x^3 + e^x \, dx$ as a limit of Riemann sums (do not evaluate this limit).

Solution: By definition, $\int_a^b f(x) := \lim_{N\to\infty} \sum_{n=1}^{N} f(x_{N;n})\Delta_N$, where $\Delta_N := \frac{b-a}{N}$ and where $x_{N;n} := a + n\,\Delta_N$ for all $n \in [1 \ldots N]$. In this case, $a = 0$ and $b = 2$, so $\Delta_N = 2/N$ and $x_{N;n} = 2n/N$. Thus, we have

$$\int_0^2 x^3 + e^x \, dx = \lim_{N\to\infty} \sum_{n=1}^{N} \left(x_{N;n}^3 + \exp\left(x_{N;n}\right)\right) \cdot \Delta_N = \boxed{\lim_{N\to\infty} \frac{2}{N} \sum_{n=1}^{N} \left[\left(\frac{2n}{N}\right)^3 + \exp\left(\frac{2n}{N}\right)\right]}.$$

(It would also be correct to sum from $n = 0$ to $n = N - 1$. More generally, it would be correct to define the integral using any 'sample points' $x_{N,1}^*, x_{N,2}^*, \ldots, x_{N,N}^*$ such that $x_{N,n}^* \in [x_{N,n-1}, x_{N,n}]$ for all $n \in [1...N]$). □

(20) 4. Compute the general antiderivative of $f(x)$.

Solution: The general antiderivative is the function $F(x) = \boxed{\frac{x^4}{4} + e^x + C.}$ □

(20) 5. Compute the exact value of $\int_0^2 x^3 + e^x \, dx$.

Solution: The Fundamental Theorem of Calculus says

$$\int_0^2 x^3 + e^x = F(2) - F(0) = \left(\frac{2^4}{4} + e^2\right) - \left(\frac{0^4}{4} + e^0\right) = \frac{16}{4} + e^2 - 1 = \boxed{3 + e^2.}$$

□