Math 1100 — Calculus, HW #1 — Due Friday, 2009-11-6

Solutions

'Common mistakes' are indicated in your marked assignmend with circled numbers, e.g. ①, ②, ③, etc. These labels are explained in the remarks following the solutions to each question.

1. Show that, at any moment in time, there exist two points on the Earth's equator which are antipodal to one another (i.e. separated by 180^o longitude), but which have exactly the same temperature.

(15) (*Hint.* For all $\theta \in \mathbb{R}$, let $T(\theta)$ be the temperature at the point on the Earth's equator with longitude θ degrees. What properties does the function T have?)

Solution: The function T is continuous. Thus, the function $S(\theta) := T(\theta - 180)$ is also continuous. Thus, the function $R(\theta) := T(\theta) - S(\theta)$ is also continuous. For any $\theta \in \mathbb{R}$, observe:

$$\Big(T(\theta) = T(\theta - 180)\Big) \quad \iff \quad \Big(R(\theta) = 0\Big). \tag{1}$$

Thus, to find two antipodal points with the same temperature, it suffices to find a point $\theta \in \mathbb{R}$ such that $R(\theta) = 0$.

Now, $R(0) = T(0) - S(-180) = T(0) - S(180)$ (because -180 degrees is the same as 180 degrees). Meanwhile, $R(180) = T(180) - S(0) = -R(0)$. Thus, we have three possibilities:

- either $R(0) = 0 = R(180)$;
- or $R(0) < 0 < R(180)$;
- or $R(0) > 0 > R(180)$.

If $R(0) = 0 = R(180)$, then we're done by statement (1). If $R(0) < 0 < R(180)$, or if $R(0) > 0 > R(180)$, then the Intermediate Value Theorem says there is some $\theta \in (0, 180)$ such that $R(\theta) = 0$ —again we're done, by statement (1).

① Some people argued that $T(\theta) = \sin(\theta)$, because this function describes the amount of sunlight hitting the earth's surface at longitude θ. This reasoning is incorrect, because temperature is *not* just equal to the current amount of incoming sunlight. Temperature is affected by the amount of heat stored in the ground, water and air, and also by heat carried by winds and ocean currents.

Nevertheless, this was an intelligent interpretation of the question, and it generally got 8/15.

More fundamentally, you can see that the above mathematical proof has nothing to do with temperature *or* sunlight *per se*. The exact same formal proof would apply if $T(\theta)$ =barometric pressure, or $T(\theta)$ = altitude, or even $T(\theta)$ = microbial population density.

② Some people observed that $T(\theta)$ must be a periodic function. This is certainly true. However, these people then vaguely argued that this immediately implies the answer. It doesn't.

Furthermore, some people argued, "since $T(\theta)$ is periodic, it must be $\sin(\theta)$ or $\cos(\theta)$." This is totally false. There are infinitely many periodic functions which look nothing like sin or cos.

③ There was a misprint in the original version of this question; I wrote 'latitude' when I meant 'longitude'. Almost everyone entirely overlooked this point, but a couple of people correlated latitude with temperature (i.e. further north = colder) and then went on to make a mistake similar to mistake #1. $\square$

2. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $|g(x)| \leq x^2$ for all $x \in \mathbb{R}$.

(5) (a) Show that $g(0) = 0$.

Solution: By hypothesis, $0 \leq |g(0)| \leq 0^2 = 0$. Thus, $|g(0)| = 0$. Thus, $g(0) = 0$.

⑤ Some people thought this question was an application of the Squeeze Theorem, and proved that $\lim_{x\to 0} g(x) = 0$. This is true. However, these people then concluded from this that $g(0) = 0$. This does *not* follow, unless you know that g is continuous at zero. No one bothered to check this, so they only got 4/5. $\square$

(15) (b) Show that $g'(0) = 0$.

Solution: We have

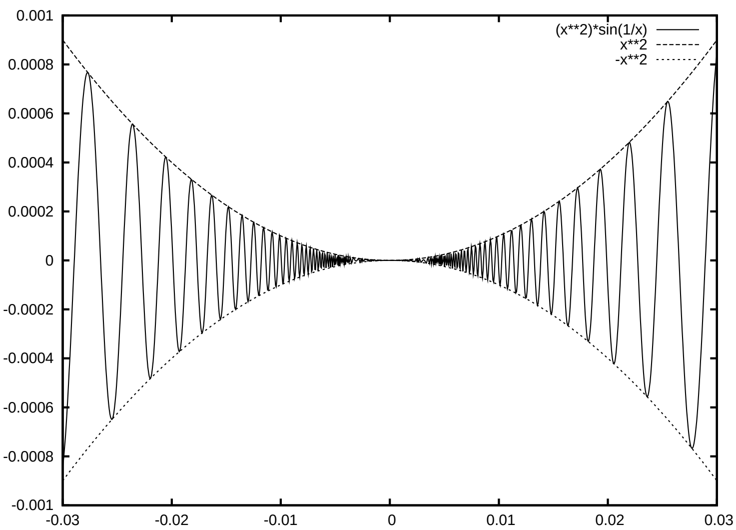
$$\begin{aligned} |g'(0)| &= \left| \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} \right| \underset{(*)}{=} \left| \lim_{x\to 0} \frac{g(x)}{x} \right| = \lim_{x\to 0} \left| \frac{g(x)}{x} \right| \\ &= \lim_{x\to 0} \frac{|g(x)|}{|x|} \underset{(\dagger)}{\leq} \lim_{x\to 0} \frac{x^2}{|x|} \underset{(\ddagger)}{=} \lim_{x\to 0} \frac{|x|^2}{|x|} \\ &= \lim_{x\to 0} |x| = 0, \end{aligned}$$

as desired. Here, $(*)$ is by part (a), and $(\dagger)$ is because $|g(x)| \leq x^2$ for all $x \in \mathbb{R}$. Finally, $(\ddagger)$is because $x^2 = |x|^2$.

④ Some people tried to apply a 'Squeeze theorem' argument to the derivative. They argued as follows: if $f(x) = -x^2$ and $h(x) = x^2$, then $f'(x) = -2x$ and $h'(x) = 2x$. They then argued:

"If $f(x) \leq g(x) \leq h(x)$,
then $f'(x) \leq g'(x) \leq h'(x)$, "

and proceeded from there. The problem is that this statement is false, as the next picture shows:



Here you can see that $-x^2 \leq g(x) \leq x^2$, however, clearly it is false that $-2x \leq g'(x) \leq 2x$, because in fact $g'(x)$ oscillates all over the place.

Another variation of the same argument was "$|g(x)| \leq x^2$, therefore $|g'(x)| \leq 2x$. Again, this is false.

People who tried these arguments generally got 7/15. Some people argued even more fallaciously that in fact $|g'(x)| = 2x$ or something like that. They got 5/15. □

(20) 3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function, and let $a \neq 0$. Suppose f is differentiable at a. Evaluate the following limit:

$$\lim_{x \to a} \frac{f(x) - f(a)}{\sqrt[3]{x} - \sqrt[3]{a}}.$$

Express your answer in terms of $f'(a)$.

Solution: Let $b := \sqrt[3]{a}$; then $a = b^3$, so $f(a) = f(b^3)$. Let $g(x) := x^3$ for all $x \in \mathbb{R}$. Let $h := f \circ g$. Then

$$\lim_{x \to a} \frac{f(x) - f(a)}{\sqrt[3]{x} - \sqrt[3]{a}} \underset{(*)}{=} \lim_{y \to b} \frac{f(y^3) - f(b^3)}{y - b} = \lim_{y \to b} \frac{f(g(y)) - f(g(b))}{y - b} \underset{(\dagger)}{=} \lim_{y \to b} \frac{h(y) - h(b)}{y - b}$$
$$\underset{(\ddagger)}{=} h'(b) \underset{(\S)}{=} f'[g(b)] \cdot g'(b) \underset{(\diamond)}{=} f'(b^3) \cdot 3b^2 = \boxed{f'(a) \cdot 3a^{2/3}.}$$

Here, $(*)$ is the change of variables $y := \sqrt[3]{x}$, so that $x = y^3$ and $f(x) = f(y^3)$. Next, $(\dagger)$ is because $h := f \circ g$. Next, $(\ddagger)$ is just the definition of the derivative. Next, $(\S)$ is by the Chain Rule, because $h := f \circ g$. Finally, $(\diamond)$ is because $g(x) = x^3$, so $g'(x) = 3x^2$.

Remark. No one used this 'chain rule' proof. Almost everyone who did the question properly instead used the 'geometric series identity' to write:

$$\frac{1}{x^{1/3} - a^{1/3}} = \frac{1}{x^{1/3} - a^{1/3}} \cdot \left(\frac{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} \right) = \frac{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}}{x - a}$$

and proceeded from there. These people got full marks.

⑥ Several people botched the geometric series identity and instead wrote something like:

$$\frac{1}{x^{1/3} - a^{1/3}} = \frac{1}{x^{1/3} - a^{1/3}} \cdot \left(\frac{x^{1/3} + x^{1/3}}{x^{1/3} + a^{1/3}} \right) = \frac{x^{1/3} + a^{1/3}}{x - a}$$

This equation is false, and it leads to the wrong answer. These people got 5/20. □

4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable functions, and define $h(x) := f(x) \cdot g(x)$ for all $x \in \mathbb{R}$.

(10) (a) Prove that $h''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$.

Solution: The Leibniz Rule says $h' = f' \cdot g + f \cdot g'$. Thus,

$$h'' = (h')' = (f' \cdot g + f \cdot g')' = (f' \cdot g)' + (f \cdot g')'$$
$$\underset{(L)}{=} \left(f'' \cdot g + f' \cdot g'\right) + \left(f' \cdot g' + f \cdot g''\right) = f'' \cdot g + 2f' \cdot g' + f \cdot g'',$$

as desired. Here, (L) is a second application of the Leibniz rule. □

(10) (b) Prove that $h'''(x) = f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + f(x)g'''(x)$.

Solution: From part (a) we have $h'' = f'' \cdot g + 2f' \cdot g' + f \cdot g''$. Thus,

$$\begin{aligned} h''' &= (h'')' = \left(f'' \cdot g + 2f' \cdot g' + f \cdot g''\right)' \\ &= \left(f'' \cdot g\right)' + \left(2f' \cdot g'\right)' + \left(f \cdot g''\right)' \\ &\underset{(L)}{=} \left(f''' \cdot g + f'' \cdot g'\right) + \left(2f'' \cdot g' + 2f' \cdot g''\right) + \left(f' \cdot g'' + f \cdot g'''\right) \\ &= f'''g + 3f''g' + 3f'g'' + fg''', \end{aligned}$$

as desired. Here, (L) is another application of the Leibniz rule. □

(25) (c) Prove that the following is true for all $n \in \mathbb{N}$:

$$h^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) \cdot g^{(k)}(x).$$

Here, $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ is the *binomial coefficient*, and this equation could also be written:

$$h^{(n)} = \binom{n}{0} f^{(n)}(x)g^{(0)}(x) + \binom{n}{1} f^{(n-1)}(x)g^{(1)}(x) + \binom{n}{2} f^{(n-2)}(x)g^{(2)}(x) + \cdots$$
$$\cdots\cdots + \binom{n}{n} f^{(0)}(x)g^{(n)}(x).$$

Hint. Use *Proof by induction*. The 'Base case' ($n = 2$) is just the Leibniz product rule. Now, let $m \in \mathbb{N}$ be arbitrary, and *assume* the result is true for $n = m$, and use this assumption, along with the Leibniz rule, to establish the result for $n = m + 1$. You will also need to use *Pascal's Identity:*

$$\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$$

Solution: Suppose the result is true for $n = m$. That is, we have:

$$h^{(m)} = \sum_{k=0}^{m} \binom{m}{k} f^{(m-k)}(x) \cdot g^{(k)}(x).$$

Then

$$\begin{aligned} h^{(m+1)}(x) &= (h^{(m)})'(x) = \sum_{k=0}^{m} \binom{m}{k} \left(f^{(m-k)} \cdot g^{(k)}\right)'(x) \\ &= \sum_{k=0}^{m} \binom{m}{k} \left((f^{(m-k)})'(x) \cdot g^{(k)}(x) + f^{(m-k)}(x) \cdot (g^{(k)})'(x)\right) \\ &= \sum_{k=0}^{m} \binom{m}{k} \left(f^{(m-k+1)}(x) \cdot g^{(k)}(x) + f^{(m-k)}(x) \cdot g^{(k+1)}(x)\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{m} \binom{m}{k} f^{(m+1-k)}(x)\cdot g^{(k)}(x) \;+\; \sum_{k=0}^{m} \binom{m}{k} f^{(m-k)}(x)\cdot g^{(k+1)}(x) \\
&\underset{(*)}{=} \sum_{k=0}^{m} \binom{m}{k} f^{(m+1-k)}(x)\cdot g^{(k)}(x) \;+\; \sum_{j=1}^{m+1} \binom{m}{j-1} f^{(m-j+1)}(x)\cdot g^{(j)}(x) \\
&= \binom{m}{0} f^{(m+1)}(x)\cdot g^{(0)}(x) \;+\; \sum_{k=1}^{m} \binom{m}{k} f^{(m+1-k)}(x)\cdot g^{(k)}(x) \\
&\qquad + \sum_{j=1}^{m} \binom{m}{j-1} f^{(m+1-j)}(x)\cdot g^{(j)}(x) \;+\; \binom{m}{m} f^{(0)}(x)\cdot g^{(m+1)}(x) \\
&\underset{(\dagger)}{=} f^{(m+1)}(x)\cdot g^{(0)}(x) + \sum_{k=1}^{m} \left(\binom{m}{k} + \binom{m}{k-1} \right) f^{(m+1-k)}(x)\cdot g^{(k)}(x) + f^{(0)}(x)\cdot g^{(m+1)}(x) \\
&\underset{(\triangle)}{=} \binom{m+1}{0} f^{(m+1)}(x)\cdot g^{(0)}(x) \;+\; \sum_{k=1}^{m} \binom{m+1}{k} f^{(m+1-k)}(x)\cdot g^{(k)}(x) \\
&\qquad + \binom{m+1}{m+1} f^{(0)}(x)\cdot g^{(m+1)}(x) \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} f^{(m+1-k)}(x)\cdot g^{(k)}(x),
\end{aligned}
$$

as desired. Here, in step $(*)$ we perform the change of variables $j := k+1$, so that $k = j-1$ and $m-k = m-j+1$ in the second sum. In step $(\dagger)$ we use the fact that $\binom{m}{0} = 1 = \binom{m}{m}$.

In step $(\triangle)$ we apply Pascal's identity; we also use the fact that $\binom{m+1}{0} = 1 = \binom{m+1}{m+1}$.

By induction, the theorem is true for all $n \in \mathbb{N}$.

⑦ Some people verified a special case of this result (e.g. the case $n = 3$ or $n = 4$) and thought that was sufficient. A verification of a special case is *not* sufficient to prove the general result. It is more like an 'experimental test'. Math is not an empirical science. An experimental test is not a mathematical proof. □