MATH 1101Y 2009 Quiz 19 (a)

1. Determine whether the series is convergent or divergent.

   (a) (2 pts)

$$\sum_{k=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Solution: We use the Integral Test:

$$\begin{aligned}
& \int_2^{\infty} \frac{1}{x(\ln x)^2} dx \\
&= \lim_{b\to\infty} \int_2^{b} \frac{1}{x(\ln x)^2} dx \\
(\text{Let } u \text{ be } \ln x.) \quad &= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u^2} \\
&= \lim_{b\to\infty} \left[-\frac{1}{u}\right]_{\ln 2}^{\ln b} \\
&= \lim_{b\to\infty} -\left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.
\end{aligned}$$

Therefore, the series is convergent. □

   (b) (1 pts)

$$\sum_{n=1}^{\infty} \frac{n^2+3n+1}{n^3+4n^2-3}$$

Solution: We use the limit form of the Comparison Test to compare this series with the series

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n}$$

Since

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{n^2+3n+1}{n^3+4n^2-3}}{\frac{n^2}{n^3}} &= \lim_{n\to\infty} \frac{n^2+3n+1}{n^3+4n^2-3} \cdot \frac{n^3}{n^2} \\
&= \lim_{n\to\infty} \frac{1+\frac{3}{n}+\frac{1}{n^2}}{1+\frac{4}{n}-\frac{3}{n^3}} = 1
\end{aligned}$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, the series is divergent. □

   (c) (2 pts)

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$$

Solution: We apply the Alternating Series Test:

We have

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0,$$

so

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Also

$$\begin{aligned} \left(\frac{\ln x}{x}\right)' &= \frac{\frac{1}{x} \cdot x - \ln x}{x^2} \\ &= \frac{1 - \ln x}{x^2} < 0 \end{aligned}$$

whenever $x > e$, $\frac{\ln n}{n}$ is decreasing for all $n \geq 3$. Therefore the series

$$\sum_{n=3}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$$

is convergent. The series

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$$

is also convergent. □