MATH 1101Y 2009 Quiz 13 (b)

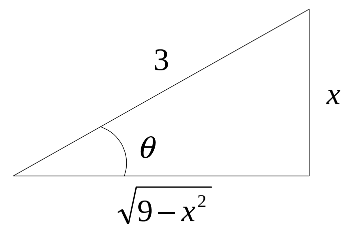
Evaluate the integral.

1. (2 pts) $\int \frac{x}{\sqrt{9-x^2}}dx$

Solution 1: Let $x = 3\sin\theta$. $dx = 3\cos\theta d\theta$.

$$\begin{aligned} \sqrt{9-x^2} &= \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta} \\ &= 3\sqrt{\cos^2\theta} = 3\cos\theta. \end{aligned}$$

$$\begin{aligned} \int \frac{x}{\sqrt{9-x^2}}dx &= \int \frac{3\sin\theta 3\cos\theta}{3\cos\theta}d\theta = 3\int \sin\theta d\theta \\ &= -3\cos\theta + C = -\sqrt{9-x^2} + C. \end{aligned}$$



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Solution 2: Let $u = 9 - x^2$. $du = -2xdx$.

$$\begin{aligned} \int \frac{x}{\sqrt{9-x^2}}dx &= -\frac{1}{2}\int \frac{-2x}{\sqrt{9-x^2}}dx = -\frac{1}{2}\int \frac{du}{\sqrt{u}} \\ &= -\frac{1}{2}2u^{\frac{1}{2}} + C = -\sqrt{9-x^2} + C. \end{aligned}$$

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2. (3 pts) $\int x^3\sqrt{1+4x^2}dx$

Solution 1: Let $x = \frac{1}{2}\tan\theta$. $dx = \frac{1}{2}\sec^2\theta d\theta$.

$$\begin{aligned} \sqrt{1+4x^2} &= \sqrt{1+4\left(\frac{1}{2}\tan\theta\right)^2} = \sqrt{1+\tan^2\theta} \\ &= \sqrt{\sec^2\theta} = \sec\theta. \end{aligned}$$

$$\begin{aligned} \int x^3\sqrt{1+4x^2}dx &= \int \left(\frac{1}{2}\right)^3 \tan^3\theta \sec\theta \frac{1}{2}\sec^2\theta d\theta \\ &= \frac{1}{16}\int \tan^3\theta\sec^3\theta d\theta \text{ (Let } u = \sec\theta.\ du = \tan\theta\sec\theta.) \\ &= \frac{1}{16}\int \tan^2\theta\sec^2\theta\tan\theta\sec\theta d\theta = \frac{1}{16}\int \left(\sec^2\theta - 1\right)\sec^2\theta\tan\theta\sec\theta d\theta \\ &= \frac{1}{16}\int \left(u^2-1\right)u^2 du = \frac{1}{16}\int \left(u^4-u^2\right)du = \frac{1}{16}\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C \\ &= \frac{1}{16}\left(\frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3}\right) + C = \frac{1}{16}\left(\frac{(1+4x^2)^{\frac{5}{2}}}{5} - \frac{(1+4x^2)^{\frac{3}{2}}}{3}\right) + C. \end{aligned}$$

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Solution 2: Let $u = 1 + 4x^2$. $du = 8xdx$. $x^2 = \frac{u-1}{4}$.

$$\begin{aligned} \int x^3\sqrt{1+4x^2}dx &= \frac{1}{8}\int x^2\sqrt{1+4x^2}\,(8x)\,dx \\ &= \frac{1}{8}\int \frac{u-1}{4}\sqrt{u}du = \frac{1}{32}\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du \\ &= \frac{1}{32}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C \\ &= \frac{1}{16}\left(\frac{1}{5}\left(1+4x^2\right)^{\frac{5}{2}} - \frac{1}{3}\left(1+4x^2\right)^{\frac{3}{2}}\right) + C. \end{aligned}$$

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