MATH 1101Y 2009 Quiz 13 (a)

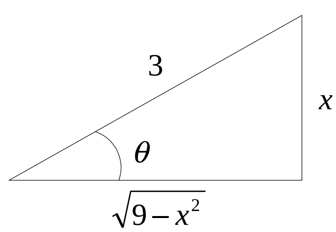
Evaluate the integral.

1. (2 pts) $\int x\sqrt{9-x^2}dx$

Solution 1: Let $x = 3\sin\theta$. $dx = 3\cos\theta d\theta$.

$$\begin{aligned}\sqrt{9-x^2} &= \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta} \\ &= 3\sqrt{\cos^2\theta} = 3\cos\theta.\end{aligned}$$

$$\begin{aligned}\int x\sqrt{9-x^2}dx &= \int 3\sin\theta 3\cos\theta 3\cos\theta d\theta \\ &= 27\int \sin\theta\cos^2\theta d\theta \text{ (Let } u = \cos\theta.\ du = -\sin\theta d\theta.) \\ &= -27\int u^2 du = -27\frac{u^3}{3} + C = -9\left(\cos^3\theta\right) + C \\ &= -9\frac{(9-x^2)^{\frac{3}{2}}}{3^3} + C = -\frac{1}{3}\left(9-x^2\right)^{\frac{3}{2}} + C.\end{aligned}$$



□

Solution 2: Let $u = 9 - x^2$. $du = -2xdx$.

$$\begin{aligned}\int x\sqrt{9-x^2}dx &= -\frac{1}{2}\int(-2x)\sqrt{9-x^2}dx \\ &= -\frac{1}{2}\int\sqrt{u}du = -\frac{1}{2}\frac{2}{3}u^{\frac{3}{2}} + C \\ &= -\frac{1}{3}\left(9-x^2\right)^{\frac{3}{2}} + C.\end{aligned}$$

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2. (3 pts) $\int \frac{x^3}{\sqrt{1+x^2}}dx$

Solution 1: Let $x = \tan\theta$. $dx = \sec^2\theta d\theta$.

$$\begin{aligned}\sqrt{1+x^2} &= \sqrt{1+\tan^2\theta} \\ &= \sqrt{\sec^2\theta} = \sec\theta.\end{aligned}$$

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1+x^2}}dx &= \int \frac{\tan^3\theta\sec^2\theta}{\sec\theta}d\theta \\
&= \int \tan^3\theta\sec\theta d\theta \text{ Let } u=\sec\theta.\ du=\tan\theta\sec\theta. \\
&= \int \tan^2\theta\tan\theta\sec\theta d\theta = \int \left(\sec^2\theta-1\right)\tan\theta\sec\theta d\theta \\
&= \int \left(u^2-1\right)du = \frac{u^3}{3}-u+C \\
&= \frac{\sec^3\theta}{3}-\sec\theta+C = \frac{(1+x^2)^{\frac{3}{2}}}{3}-\sqrt{1+x^2}+C.
\end{aligned}$$

□

Solution 2: Let $u = 1 + x^2$. $du = 2xdx$. $x^2 = u - 1$.

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1+x^2}}dx &= \frac{1}{2}\int \frac{x^2(2x)\,dx}{\sqrt{1+x^2}} = \frac{1}{2}\int \frac{u-1}{\sqrt{u}}du \\
&= \frac{1}{2}\int \left(\sqrt{u}-\frac{1}{\sqrt{u}}\right)du = \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right)+C \\
&= \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}-\sqrt{1+x^2}+C.
\end{aligned}$$

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