MATH 1101Y 2009 Quiz 11 (b)

1. (2 pts) Find the area of the region enclosed by the curves $y = -x^2 + 3x + 2$ and $y = 2x$.

   Solution: We first find the intersection of these curves. Let

$$\begin{aligned} -x^2 + 3x + 2 &= 2x \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \end{aligned}$$

$$x = -1 \text{ or } x = 2.$$

   Let $x = 0$. $-x^2 + 3x + 2 = 2$ and $2x = 0$. The curve $-x^2 + 3x + 2$ is on top. The area is

$$\begin{aligned} & \int_{-1}^{2} \left(-x^2 + 3x + 2 - 2x\right) dx \\ = & \int_{-1}^{2} \left(-x^2 + x + 2\right) dx \\ = & \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{2} \\ = & \left(-\frac{2^3}{3} + \frac{2^2}{2} + 2 \cdot 2\right) - \left(-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 2(-1)\right) \\ = & \frac{9}{2}. \end{aligned}$$

$\square$

2. (3 pts) Use the method of cylindrical shells to find the volume generated by rotating the regoin bounded by the curves $y = \frac{1}{1+(x-2)^2}, y = 0, x = 1$ and $x = 3$ about the y-axis.

   Solution: Using the method of cylindrical shells, we have

$$V = \int_1^3 2\pi x \frac{1}{1+(x-2)^2} dx$$

   (Let $u = x - 2$, $x = u + 2$, $du = dx$, $x = 3 \to u = 1$, $x = 1 \to u = -1$.)

$$\begin{aligned} &= 2\pi \int_{-1}^{1} \frac{u+2}{1+u^2} du \\ &= 2\pi \int_{-1}^{1} \frac{u}{1+u^2} du + 2\pi \int_{-1}^{1} \frac{2}{1+u^2} du \end{aligned}$$

   (Let $v = 1 + u^2$. $dv = 2udu$. $u = -1 \to v = 2$. $u = 1 \to v = 2$.)

$$\begin{aligned} &= \pi \int_2^2 \frac{dv}{v} + 4\pi \left[\tan^{-1} u\right]_{-1}^{1} \\ &= 0 + 4\pi \left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) \\ &= 2\pi^2. \end{aligned}$$

$\square$