MATH 1101 2009 Assignment 3

Due Feb. 26, 2010

Show all your work.

1. Evaluate the integral. (2.5 pts each)

   (a) $\int \cos^3 x \tan^2 x dx$

   Solution:

$$\begin{aligned} & \int \cos^3 x \tan^2 x dx \\ = & \int \cos^3 x \frac{\sin^2 x}{\cos^2 x} dx = \int \sin^2 x \cos x dx \\ = & \int u^2 du \ (u = \sin x,\ du = \cos x dx) \\ = & \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C. \end{aligned}$$

$\square$

   (b) $\int t \sin\left(\sqrt{t}\right) dt$

   Solution:

$$\begin{aligned} & \int t \sin\left(\sqrt{t}\right) dt \\ = & \int x^2 \sin x \,(2x)\, dx \ (t = x^2,\ dt = 2x dx) \\ = & 2 \int x^3 \sin x dx = 2\left(x^3(-\cos x) - \int 3x^2(-\cos x)\, dx\right) \\ = & -2x^3 \cos x + 6 \int x^2 \cos x dx \\ = & -2x^3 \cos x + 6\left(x^2 \sin x - \int 2x \sin x dx\right) \\ = & -2x^3 \cos x + 6x^2 \sin x - 12 \int x \sin x dx \\ = & -2x^3 \cos x + 6x^2 \sin x - 12\left(x(-\cos x) - \int (-\cos x)\, dx\right) \\ = & -2x^3 \cos x + 6x^2 \sin x + 12x \cos x + 12 \sin x + C \\ = & -2\left(\sqrt{t}\right)^3 \cos\left(\sqrt{t}\right) + 6\left(\sqrt{t}\right)^2 \sin\left(\sqrt{t}\right) + 12\left(\sqrt{t}\right) \cos\left(\sqrt{t}\right) + 12 \sin\left(\sqrt{t}\right) + C. \end{aligned}$$

$\square$

   (c) $\int \frac{1}{\sqrt{y^2-4y+5}} dy$

Solution:

$$\begin{aligned}
&\int \frac{1}{\sqrt{y^2-4y+5}}dy \\
= &\int \frac{1}{\sqrt{y^2-4y+4+1}}dy = \int \frac{1}{\sqrt{(y-2)^2+1}}dy \\
= &\int \frac{1}{\sqrt{u^2+1}}du \text{ (Let } u = y-2.\ du = dy.) \\
= &\int \frac{1}{\sec\theta}\sec^2\theta d\theta \text{ (Let } u = \tan\theta.\ du = \sec^2\theta d\theta.) \\
= &\int \sec\theta d\theta = \ln(\tan\theta + \sec\theta) + C \\
= &\ln\left(u + \sqrt{u^2+1}\right) + C = \ln\left(y-2+\sqrt{(y-2)^2+1}\right) + C \\
= &\ln\left(y-2+\sqrt{y^2-4y+5}\right) + C.
\end{aligned}$$

□

(d) $\int \frac{-2x^2+8x-12}{x^3+4x}dx$

Solution: Using partial fractions, we have

$$\begin{aligned}
\frac{-2x^2+8x-12}{x^3+4x} &= \frac{-2x^2+8x-12}{x(x^2+4)} \\
&= \frac{x}{x^2+4} + \frac{8}{x^2+4} - \frac{3}{x}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int \frac{-2x^2+8x-12}{x^3+4x}dx \\
= &\int \frac{x}{x^2+4}dx + \int \frac{8}{x^2+4}dx - \int \frac{3}{x}dx \\
= &\frac{1}{2}\int \frac{2x}{x^2+4}dx + 8\int \frac{1}{x^2+4}dx - 3\int \frac{1}{x}dx \\
= &\frac{1}{2}\int \frac{du}{u} + 8\int \frac{2\sec^2\theta d\theta}{(2\tan\theta)^2+4} - 3\ln|x| \\
\text{(where } u = &\ x^2+4 \text{ in the first integral}, x = 2\tan\theta \text{ in the second)} \\
= &\frac{1}{2}\ln|u| + 16\int \frac{\sec^2\theta d\theta}{4\sec^2\theta} - 3\ln|x| \\
= &\frac{1}{2}\ln\left|x^2+4\right| + 4\theta - 3\ln|x| + C \\
= &\frac{1}{2}\ln\left|x^2+4\right| + 4\tan^{-1}\left(\frac{x}{2}\right) - 3\ln|x| + C.
\end{aligned}$$

□

(e) $\int \frac{dx}{x+\sqrt[3]{x}}$

Solution:

$$\begin{aligned}
&\int \frac{dx}{x+\sqrt[3]{x}} \\
=& \int \frac{3t^2}{t^3+t}dt \text{ (Let } x = t^3.\ dx = 3t^2dt) \\
=& \int \frac{3t}{t^2+1}dt = \frac{3}{2}\int \frac{2t}{t^2+1}dt \\
=& \frac{3}{2}\int \frac{du}{u} \text{ (Let } u = t^2+1.\ du = 2tdt.) \\
=& \frac{3}{2}\ln|u| + C = \frac{3}{2}\ln\left|t^2+1\right| + C \\
=& \frac{3}{2}\ln\left|x^{\frac{2}{3}}+1\right| + C.
\end{aligned}$$

□

(f) $\int \sqrt{e^x+1}dx$

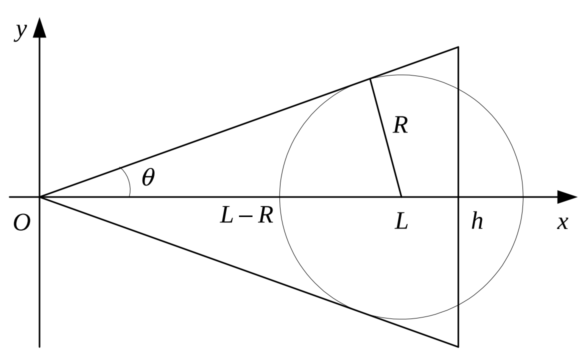
Solution:

$$\begin{aligned}
&\int \sqrt{e^x+1}dx \\
=& \int \frac{\sqrt{e^x+1}\cdot e^x}{e^x}dx \text{ (Let } u = e^x+1.\ e^x = u-1.\ du = e^xdx.) \\
=& \int \frac{\sqrt{u}}{u-1}du \text{ (Let } u = v^2.\ du = 2vdv.) \\
=& \int \frac{v}{v^2-1}2vdv = \int \frac{2v^2}{v^2-1}dv = 2\int\left(1+\frac{1}{v^2-1}\right)dv \\
=& 2v + \int\left(\frac{1}{v-1}-\frac{1}{v+1}\right)dv = 2v + \ln|v-1| - \ln|v+1| + C \\
=& 2v + \ln\left(\frac{v-1}{v+1}\right) + C = 2\sqrt{u} + \ln\left(\frac{\sqrt{u}-1}{\sqrt{u}+1}\right) + C \\
=& 2\sqrt{e^x+1} + \ln\left(\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right) + C.
\end{aligned}$$

□

2. (5 pts) Page 449 #10. A paper drinking cup filled with water has the shape of a cone with height h and semivertical angle θ. A ball is placed carefully in the cup, thereby displacing some of the water and making it overflow. What is the radius of the ball that causes the greatest volume of water to spill out of the cup?

Solution:



As in the diagram, we place the cone horizontally with its tip at the origin. Let the radius of the ball be R and the x-coordinate of its centre be L. The volume of displaced water is the volume of the revolution that is obtained by rotating the upper part of the circle from $x = L - R$ to $x = h$ about the x-axis. Since the edge of the cone is tangent to the circle, we have $\frac{R}{L} = \sin\theta$, or $R = L\sin\theta$.

The equation of the circle is $(x-L)^2 + y^2 = R^2$ and the upper half is the graph of the function

$$y = \sqrt{R^2 - (x-L)^2}.$$

Using the washer method, we find the volume of displaced water to be

$$\begin{aligned}
V &= \int_{L-R}^{h} \pi \left(\sqrt{R^2 - (x-L)^2}\right)^2 dx \\
&= \pi \int_{L-R}^{h} \left(R^2 - (x-L)^2\right) dx \\
&= \pi \left[R^2 x - \frac{(x-L)^3}{3}\right]_{L-R}^{h} \\
&= \pi \left(R^2 h - \frac{(h-L)^3}{3} - R^2(L-R) + \frac{(-R)^3}{3}\right) \\
&= \pi \left((L\sin\theta)^2 h - \frac{(h-L)^3}{3} - (L\sin\theta)^2 (L - L\sin\theta) - \frac{(L\sin\theta)^3}{3}\right) \\
&= \pi \left(L^2 \sin^2\theta h - \frac{(h-L)^3}{3} - L^3 \sin^2\theta (1 - \sin\theta) - \frac{L^3 \sin^3\theta}{3}\right) \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\frac{dV}{dL} &= \pi \left(2Lh\sin^2\theta + (h-L)^2 - 3L^2 \sin^2\theta (1-\sin\theta) - L^2 \sin^3\theta\right) \\
&= \pi \left(2Lh\sin^2\theta + (h-L)^2 - 3L^2\sin^2\theta + 3L^2\sin^3\theta - L^2\sin^3\theta\right) \\
&= \pi \left((h-L)^2 \underline{-3L^2\sin^2\theta} + 2Lh\sin^2\theta + 2L^2\sin^3\theta\right) \\
&= \pi \left((h-L)^2 \underline{-L^2\sin^2\theta - 2L^2\sin^2\theta} + 2Lh\sin^2\theta + 2L^2\sin^3\theta\right) \\
&= \pi \left(\left[(h-L)^2 - L^2\sin^2\theta\right] + \left[-2L^2\sin^2\theta + 2Lh\sin^2\theta + 2L^2\sin^3\theta\right]\right) \\
&= \pi \left((L - h + L\sin\theta)\left(\underline{L - h - L\sin\theta}\right) - 2L\sin^2\theta \left(\underline{L - h - L\sin\theta}\right)\right) \\
&= \pi (L - h - L\sin\theta)\left(L - h + L\sin\theta - 2L\sin^2\theta\right).
\end{aligned}$$

$\frac{dV}{dL} = 0$ if and only if $L - h - L\sin\theta = 0$ or $L - h + L\sin\theta - 2L\sin^2\theta = 0$.

$$\begin{aligned}
L - h - L\sin\theta &= 0 \Leftrightarrow L(1 - \sin\theta) = h \\
&\Leftrightarrow L = \frac{h}{1 - \sin\theta}.
\end{aligned}$$

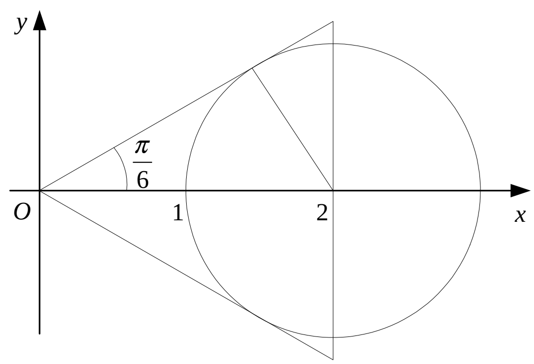
$$\begin{aligned}
L - h + L\sin\theta - 2L\sin^2\theta &= 0 \Leftrightarrow L\left(1 + \sin\theta - 2\sin^2\theta\right) = h \\
&\Leftrightarrow L = \frac{h}{1 + \sin\theta - 2\sin^2\theta}.
\end{aligned}$$

If $L = \frac{h}{1-\sin\theta}$, substituting that into (1), we have

$$
\begin{aligned}
V &= \pi\left(L^2\sin^2\theta h - \frac{(h-L)^3}{3} - L^3\sin^2\theta\,(1-\sin\theta) - \frac{L^3\sin^3\theta}{3}\right) \\
&= \pi\left(\left(\frac{h}{1-\sin\theta}\right)^2\sin^2\theta h - \frac{\left(h-\frac{h}{1-\sin\theta}\right)^3}{3} - \left(\frac{h}{1-\sin\theta}\right)^3\sin^2\theta\,(1-\sin\theta) - \frac{\left(\frac{h}{1-\sin\theta}\right)^3\sin^3\theta}{3}\right) \\
&= \pi\left(\frac{h^3\sin^2\theta}{(1-\sin\theta)^2} + \frac{h^3\sin^3\theta}{3\,(1-\sin\theta)^3} - \frac{h^3\sin^2\theta}{(1-\sin\theta)^2} - \frac{h^3\sin^3\theta}{3\,(1-\sin\theta)^3}\right) = 0.
\end{aligned}
$$

Therefore, at the other critical number, $L = \frac{h}{1+\sin\theta-2\sin^2\theta}$, V reaches its maximum. (There must be a maximum value of V and it must occur at a critical number.) That is, when $R = L\sin\theta = \frac{h\sin\theta}{1+\sin\theta-2\sin^2\theta}$, the maximum volume of water is displaced.

For example, if $h = 2$ and $\theta = \frac{\pi}{6}$, to maximize the volume of displaced water, we would choose $R = \frac{2\sin\frac{\pi}{6}}{1+\sin\frac{\pi}{6}-2\sin^2\frac{\pi}{6}} = 1$.



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