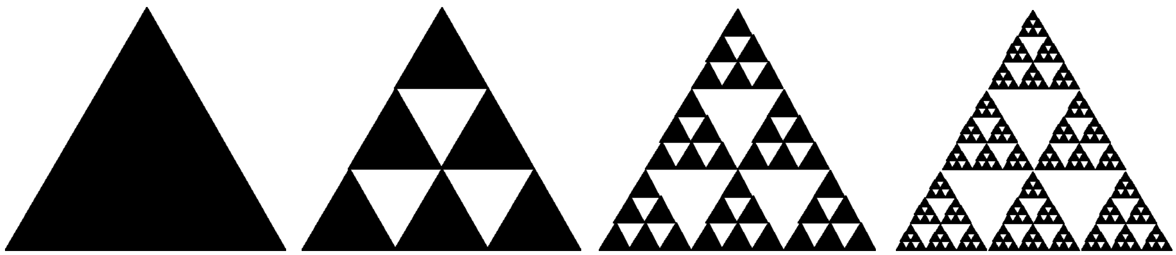
Mathematics 110 – Calculus of one variable
Trent University 2003-2004

SOLUTIONS TO ASSIGNMENT #1

Fractal nonsense or nonsense fractal?

Start with an equilateral triangle of area one, set up so its base is horizontal. At step one, divide it up into nine equal equilateral subtriangles and remove the (insides of) three that point downwards. At step two, do the same to each of the six surviving subtriangles. At step three, do the same to each of the thirty six surviving subsubtriangles. Here's a picture:



Now just keep on going! The basic problem is to figure out what the shape that is left after infinitely many steps is like. (Something *is* left over. For example, the corners of the original triangle, of the subtriangles, of the subsubtriangles, *etc.*, never get removed.) What one has to work with is understanding the process and the fact that the process generates ever closer approximations of the "final" shape.

1. What is the area of the final shape? *[2]*

Solution. At first, call it step 0, we have an equilateral triangle of area $a_0 = 1$; at step 1, we remove 3 of 9 equal subtriangles, leaving an area of $a_1 = \frac{2}{3}$; at step 2, we remove 3 of 9 equal subsubtriangles from each of the remaining subtriangles, leaving an area of $a_2 = \frac{2}{3} \cdot \frac{2}{3} = \left(\frac{2}{3}\right)^2$; ... ; at step n, we remove 3 of 9 equal sub^{n}triangles from each of the remaining sub^{n-1}triangles, leaving an area of $a_n = \left(\frac{2}{3}\right)^n$; and so on.

The area of the final shape is then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0\,.$$

(If it's not obvious, check §11.1 to see why this limit is 0.) ■

2. What is the length of the border of the final shape? *[2]*
[The borders inside the shape do count!]

Solution. At first, call it step 0, we have an equilateral triangle of some border length $p_0{}^\dagger$; at step 1, we remove 3 of 9 equal subtriangles, leaving a combined border length of $p_1 = 6 \cdot \frac{1}{3} \cdot p_0 = 2p_0$ for the remaining subtriangles; at step 2, we remove 3 of 9 equal subsubtriangles from each of the remaining subtriangles, leaving a combined border length of $p_2 = 6 \cdot \frac{1}{3} \cdot p_1 = 2^2 p_0$; ... ; at step n, we remove 3 of 9 equal sub^{n}triangles from each of the remaining sub^{n-1}triangles, leaving a combined border length of $p_n = 6 \cdot \frac{1}{3} \cdot p_{n-1} = 2^n p_0$; and so on.

The border length of the final shape is then

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} 2^n p_0 = \infty\,.$$

† It doesn't really matter what p_0 is, so long as it's positive. If you're curious, $p_0 = \frac{2}{3^{1/4}}$.

(If it's not obvious, check §11.1 to see why this limit is ∞.) ■

3. Considering your answers to **1** and **2**, can the shape in question be real? Why or why not? *[1]*

Solution. This is a really a problem in philosophy. Depending on the assumptions and reasoning one brings to the question, the answer could legitimately be argued to be "yes," "no," "maybe," "I don't know," ...

For one of your instructors' two cents worth: If you accept the set of all real numbers as being "real," you're probably going to be stuck with accepting the shape in question is being real. (Why?) As soon as one accepts the completed infinities and/or infinite processess necessary to assemble infinite sets [which said instructor does], it's hard to escape having some paradoxical objects in one's mathematical reality. ■

Limitations

It's pretty obvious that $\lim_{t \to +\infty} \frac{1}{t} = 0$ and that $\lim_{u \to 0^+} \frac{1}{u} = +\infty$. This relationship, together with the following fact,

4. Suppose f is some function whose domain includes the interval $(0, c)$, where c is some constant greater than 0. Use the $\varepsilon - \delta$ and $\varepsilon - N$ definitions of limits to show that $\lim_{t \to +\infty} f(t) = \lim_{u \to 0^+} f\left(\frac{1}{u}\right)$. *[3]*

Solution. We may as well assume that both limits exist, because the question is a little fishy if they don't. Suppose, that L is the number for which $\lim_{t \to +\infty} f(t) = L$, *i.e.*

(*) for any $\varepsilon > 0$, there is an N, such that if $t > N$, then $-\varepsilon < f(t) - L < \varepsilon$.

We will try to show that $\lim_{u \to 0^+} f\left(\frac{1}{u}\right)$ is also equal to L. That is, we'll try to show that

for any $\varepsilon > 0$, there is an $\delta > 0$, such that if $0 < u - 0 < \delta$, then $-\varepsilon < f\left(\frac{1}{u}\right) - L < \varepsilon$.

As usual, we'll start by trying to reverse-engineer δ from the ε. Given an $\varepsilon > 0$, we need to get:

$$-\varepsilon < f\left(\frac{1}{u}\right) - L < \varepsilon$$

The problem is that we don't know anything about the function f, except that the *other* limit exists and equals L; in other words, that (*) is true of f. Plugging in $\frac{1}{u}$ for t in (*), and using the given ε, tells us that there is an N such that if $\frac{1}{u} > N$, then $-\varepsilon < f\left(\frac{1}{u}\right) - L < \varepsilon$.

We actually need a δ so that we can get $f\left(\frac{1}{u}\right)$ within ε of L whenever $0 < u < \delta$. This δ can be deduced from the N:

$$\frac{1}{u} > N \iff 1 > uN \iff \frac{1}{N} > u$$

Hence $\delta = \frac{1}{N}$ should do.

To check that $\delta = \frac{1}{N}$ works, suppose we have an $\varepsilon > 0$. Then $0 < u - 0 < \delta = \frac{1}{N}$ implies that $u < \frac{1}{N}$, so $\frac{1}{u} > N$. By (*), it follows that that $-\varepsilon < f\left(\frac{1}{u}\right) - L < \varepsilon$, as desired.

Hence $\lim_{u \to 0^+} f\left(\frac{1}{u}\right) = L$ too, so $\lim_{t \to +\infty} f(t) = \lim_{u \to 0^+} f\left(\frac{1}{u}\right)$. ■

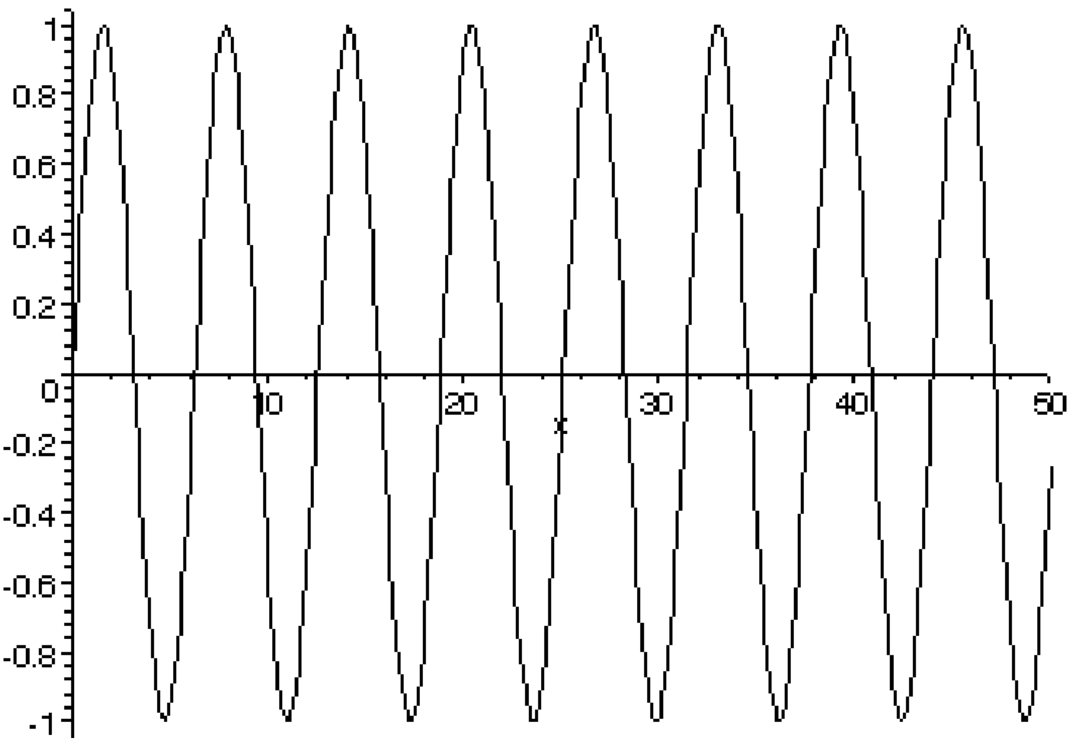
NOTE: The argument given above actually guarantees that if $\lim_{t\to+\infty} f(t)$ exists, then so does $\lim_{u\to 0^+} f\left(\frac{1}{u}\right)$. A similar argument would show that if $\lim_{u\to 0^+} f\left(\frac{1}{u}\right)$ exists, then so does $\lim_{t\to+\infty} f(t)$.

... can come in handy occasionally when faced with some otherwise messy limits.

5. Use **4** to explain why $\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right)$ does not exist. *[2]*

Solution. It does not exist because $\lim_{t\to\infty} \sin(t)$ doesn't exist. As noted above, arguments like those in the solution to **4** guarantee that if one limit exists, so does the other. It follows that if one does not exists, the other can't exist either.

Of course, the assertion that $\lim_{t\to\infty} \sin(t)$ doesn't exist needs a little justification. It is enought to observe that $\sin(t)$ continues to oscillate between -1 and 1 as $t \to \infty$:



This graph was generated by `Maple` using the command `plot(sin(t),t=0..50);` .

Since $\sin(t)$ continues to oscillate without diminishing the scale of the oscillation, or changing its frequency, $\sin(t)$ can't approach any single value as t goes off to ∞. ∎

Bonus!

2π. Suppose h is a function such that for *every* sequence a_n with $\lim_{n\to\infty} a_n = a$, it is true that $\lim_{n\to\infty} h(a_n) = h(a)$. Does h have to be continuous at a? Prove it does or find a counterexample. *[2]*

Just a hint! h does have to be continuous at a under the given assumption. It's sufficient to show – and easier than trying a direct argument – that if h is not continuous at a, there is a sequence a_n with limit a for which $h(a_n)$ does not have limit $h(a)$. Think it over! ∎