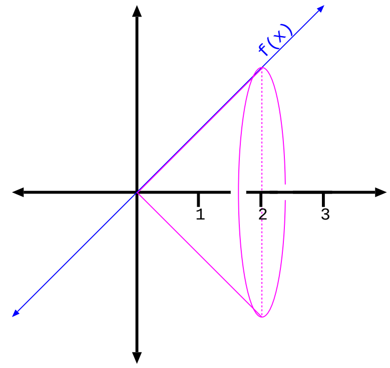
Math 110B — Final Exam — April 23, 2003

- There are 12 questions, worth **200** marks in total.
  Read all questions before beginning.
- You may use a formula sheet, or the *Formulas for Success* pamphlet.
- *Justify your answers.* Show all steps in your computations.
- Please indicate your final answer by putting a box around it.
- Please write neatly and legibly. *Illegible answers will not be graded.*
- If anything confuses you, please ask me about it.



10 — 1. Compute the following limits (5 points each):

(a) $\lim_{x \to 0} x \log(x)$. **(b)** $\lim_{x \to \infty} \dfrac{x^3 - 6x^2}{7x^3 - x + 4}$.

20 — 2. Sketch the curve of $f(x) = \log(x^2+1) - 2\arctan(x)$. Label all maxima and minima (if any) and asymptotes (if any). Find $\lim_{x \to \pm\infty} f(x)$.

5 — 3. If $f(x) = 3x + 5$, then $f(2) = 11$. How close must x be to 2, in order that $|f(x) - 11| < \frac{1}{2}$?

15 — 4. Compute $f'(x)$ in each of the following cases:

(5) (a) $f(x) = \ln\left(\dfrac{\sqrt[7]{x^2+6} \cdot (x^3-5)^2}{(x-2)^4}\right)$. (**Hint:** simplify first.)

(5) (b) $f(x) = \sin^2(x) \cdot \log(x) + \dfrac{\tan(x)}{x^3}$.

(5) (c) $f(x) = \displaystyle\int_0^{\cos(x)} t^3 \, dt$.

15 — 5. A snowball is melting. At each instant, its volume decreases at a rate directly proportional to its surface area. For example, when the surface has area $15\,\mathtt{cm}^2$, the snowball is losing water at $30\,\mathtt{cm}^3/\mathtt{sec}$. When the surface area is $10\,\mathtt{cm}^2$, the snowball is losing $20\,\mathtt{cm}^3/\mathtt{sec}$.

(2) (a) If $V(t)$ is the volume of the snowball at time t, and $A(t)$ is its surface area, write a formula for $\dfrac{dV}{dt}$ in terms of $A(t)$.

(3) (b) A sphere of radius r has volume $V(r) = \frac{4}{3}\pi r^3$ and surface area $A(r) = 4\pi r^2$. Write a formula for $\dfrac{dV}{dr}$ in terms of $A(r)$.

(5) (c) Use #5a and #5b to derive an equation for $\dfrac{dr}{dt}$.

(3) (d) Use #5c to derive an equation for r as a function of t.

(2) (e) If the snowball has a radius of $10\,\mathtt{cm}$, then how long until it completely melts?

45 6. Do *three* of the following five integrals: (15 marks each)

(a) $\displaystyle\int_{\pi/6}^{\pi/4} \cos(\theta)\csc(\theta)^5 \; d\theta.$

(b) $\displaystyle\int \frac{x}{(x^2+1)^2} \; dx.$

(c) $\displaystyle\int_0^{\pi} x\cos(5x) \; dx.$

(d) $\displaystyle\int \cos^5(\theta)\cdot\sin(\theta)^2 \; d\theta.$

(e) $\displaystyle\int \sin^2(\theta) \; d\theta.$

10 7. Determine whether or not the following improper integral converges. *Justify* your answer. (**Note:** *You do not have to compute the integral*)

$$\int_1^{\infty} \frac{\log(x)\cdot(x^3+4)}{x^6+2} \; dx.$$

15 8. Compute $\displaystyle\int_0^1 3x+7 \; dx$ using right-hand Riemann sums. (*Do not* antidifferentiate.)

10 9. Compute the volume of a cone of height 2 and radius 2, using the **Method of Disks** (**Hint:** *See picture on front*)

30 10. For each of the following series, determine whether the series **diverges**, **converges conditionally**, or **converges absolutely**.

(10) (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\log(n)+2}.$

(10) (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(n)}{\sqrt{n+1}}.$

(10) (c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^3+1}{3^n}.$

10 11. Consider power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n^2+1)}{n!}x^n \;=\; 1-2x+\frac{5}{2}x^2-\frac{10}{6}x^3+\frac{17}{24}x^4-\ldots$

(9) (a) Find the **radius of convergence** of this power series.

(1) (b) What is the **interval of convergence** of this power series?

15 12. Let $f(x)=\sin(2x)+\cos(2x)$

(5) (a) Compute $f'(x)$, $f''(x)$, $f'''(x)$ and $f''''(x)$.

(10) (b) Write the first 8 terms in the McLaurin series for $f(x)$. (Don't worry about the radius of convergence.)