# Math 110 —Assignment #6

Due: Monday, February 10

- *Justify your answers.* Show all steps in your computations.
- Please indicate your final answer by putting a $\boxed{\text{box}}$ around it.
- Please write neatly and legibly. *Illegible answers will not be graded.*
- **Math 110A:** When finished, please give your assignment to Stefan or leave it under his door.
- **Math 110B:** When finished, please place your assignment in slot marked MATH 110 in the big white box outside the Math Department Office in Lady Eaton College.

Let $f : [0, 2\pi] \longrightarrow \mathbb{R}$ be a function. For $n = 1, 2, 3, \ldots$, we define the **Fourier Coefficients**:

$$A_n \;=\; \int_0^{2\pi} f(x) \cdot \cos(nx)\, dx, \qquad \text{and} \qquad B_n \;=\; \int_0^{2\pi} f(x) \cdot \sin(nx)\, dx.$$

For example, if $f(x) = \sin^3(x)$, and $n = 7$, then

$$A_7 \;=\; \int_0^{2\pi} \sin^3(x) \cdot \cos(7x)\, dx, \qquad \text{and} \qquad B_7 \;=\; \int_0^{2\pi} \sin^3(x) \cdot \sin(7x)\, dx.$$

Physically speaking, if $f(x)$ describes the vibration of a string, then $A_7$ and $B_7$ measure the amount of energy vibrating at 7 cycles per second (ie. 7 `Hz`). Likewise, $A_8$ and $B_8$ measure the amount of energy vibrating at 8 `Hz`, etc.

Suppose $f(x) = \sin^3(x)$.

1. Compute $f'(x)$ and $f''(x)$.

**Solution:** $f'(x) = 3\sin^2(x)\cos(x)$ and $f''(x) = 6\sin(x)\cos^2(x) - 3\sin^3(x)$.

2. Show that $|f'(x)| \leq 3$ for all $x \in [0, 2\pi]$, and $|f'(x)| \leq 9$ for all $x \in [0, 2\pi]$.

**Solution:** $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$ for all $x$. Thus, $|f'(x)| = \left|3\sin^2(x)\cos(x)\right| = 3\,|\sin(x)| \cdot |\cos(x)| \leq 3 \cdot 1 \cdot 1 = 3$.

Likewise, $|f''(x)| = \left|6\sin(x)\cos^2(x) - 3\sin^3(x)\right| \leq_{(\Delta)} \left|6\sin(x)\cos^2(x)\right| + \left|3\sin^3(x)\right| = 6\,|\sin(x)| \cdot |\cos(x)|^2 + 3\,|\sin(x)|^3 \leq 6 \cdot 1 \cdot 1^2 + 3 \cdot 1^3 = 9$.

Here, $(\Delta)$ is the Triangle Inequality.

3. If $A_7$ is the Fourier coefficient defined above, show that

$$A_7 \;=\; \frac{-3}{7}\int_0^{2\pi} \sin(x)^2\cos(x)\cdot\sin(7x)\;dx.$$

(**Hint:** *Use integration by parts*). Generalize this to show that, for any $n = 1, 2, 3, \ldots$,

$$A_n \;=\; \frac{-3}{n}\int_0^{2\pi} \sin(x)^2\cos(x)\cdot\sin(nx)\,dx, \qquad \text{and} \qquad B_n \;=\; \frac{3}{n}\int_0^{2\pi} \sin(x)^2\cos(x)\cdot\cos(nx)\,dx.$$

**Solution:** We apply integration by parts. Let $f(x) = \sin^3(x)$, and suppose $g'(x) = \cos(nx)$. Thus, $f'(x) \underset{\#1}{=\!=} 3\sin^2(x)\cos(x)$ and $g(x) = \frac{1}{n}\sin(nx)$, so that

$$\begin{aligned}
A_n &= \int_0^{2\pi} \sin^3(x)\cdot\cos(nx)\;dx \quad = \quad \int_0^{2\pi} f(x)\cdot g'(x)\;dx \\
&= f(x)\cdot g(x)\,\Big|_{x=0}^{x=2\pi} \;-\; \int_0^{2\pi} f'(x)\cdot g(x)\;dx \\
&= \frac{1}{n}\sin^3(x)\cdot\sin(nx)\,\Big|_{x=0}^{x=2\pi} \;-\; \frac{1}{n}\int_0^{2\pi} 3\sin(x)^2\cos(x)\cdot\sin(nx)\;dx \\
&= \frac{1}{n}\Big(\sin^3(2\pi)\cdot\sin(2n\pi) - \sin^3(0)\cdot\sin(0)\Big) \;-\; \frac{3}{n}\int_0^{2\pi} \sin(x)^2\cos(x)\cdot\sin(nx)\;dx \\
&\underset{(P)}{=\!=} -\frac{3}{n}\int_0^{2\pi} \sin(x)^2\cos(x)\cdot\sin(nx)\;dx.
\end{aligned}$$

To see equality $(P)$, observe that $\sin^3(2\pi) = \sin^3(0)$ and $\sin(2n\pi) = \sin(0)$; hence, $\sin^3(2\pi)\cdot\sin(2n\pi) = \sin^3(0)\cdot\sin(0)$.

Likewise, if $g'(x) = \sin(nx)$, then $g(x) = \frac{-1}{n}\cos(nx)$, so that

$$\begin{aligned}
B_n &= \int_0^{2\pi} \sin^3(x)\cdot\sin(nx)\;dx \quad = \quad \int_0^{2\pi} f(x)\cdot g'(x)\;dx \\
&= f(x)\cdot g(x)\,\Big|_{x=0}^{x=2\pi} \;-\; \int_0^{2\pi} f'(x)\cdot g(x)\;dx \\
&= \frac{-1}{n}\sin^3(x)\cdot\cos(nx)\,\Big|_{x=0}^{x=2\pi} \;+\; \frac{1}{n}\int_0^{2\pi} 3\sin(x)^2\cos(x)\cdot\cos(nx)\;dx \\
&= \frac{-1}{n}\Big(\sin^3(2\pi)\cdot\cos(2n\pi) - \sin^3(0)\cdot\cos(0)\Big) \;+\; \frac{3}{n}\int_0^{2\pi} \sin(x)^2\cos(x)\cdot\cos(nx)\;dx \\
&\underset{(P)}{=\!=} \frac{3}{n}\int_0^{2\pi} \sin(x)^2\cos(x)\cdot\cos(nx)\;dx,
\end{aligned}$$

where equality $(P)$ is because $\sin^3(2\pi)\cdot\cos(2n\pi) = \sin^3(0)\cdot\cos(0)$.

4. Conclude that, for all $n = 1, 2, 3, \ldots$,

$$|A_n| \;\le\; \frac{6\pi}{n}, \qquad \text{and} \qquad |B_n| \;\le\; \frac{6\pi}{n}.$$

(For example, $A_{60} < \frac{\pi}{10}$.) Hence, there is 'little energy' in the 'high frequency' vibrations.
(**Hint:** *Do not explicitly compute any integrals. Instead, combine #2 and #3, and use the* `Comparison Properties of the Integral` *from §5.2 of the text*).

**Solution:** From #3, we know that $A_n = \frac{-1}{n}\int_0^{2\pi} f'(x)\cdot\sin(nx)\ dx$. From #2, we know that $|f'(x)| \leq 3$ for all $x \in [0, 2\pi]$. Thus,

$$
\begin{aligned}
|A_n| &= \left|\frac{-1}{n}\int_0^{2\pi} f'(x)\cdot\sin(nx)\ dx\right| &=& \frac{1}{n}\left|\int_0^{2\pi} f'(x)\cdot\sin(nx)\ dx\right| \\
&\leq_{(*)} \frac{1}{n}\int_0^{2\pi} \left|f'(x)\cdot\sin(nx)\right|\ dx &=& \frac{1}{n}\int_0^{2\pi} \left|f'(x)\right|\cdot|\sin(nx)|\ dx \\
&\leq \frac{1}{n}\int_0^{2\pi} 3\cdot 1\ dx = \frac{1}{n}6\pi &=& \frac{6\pi}{n}.
\end{aligned}
$$

Here, inequality $(*)$ is by `Comparison Property` #8 on page 387 of §5.2.

The argument for $B_n$ is the same, only with $\cos(nx)$ instead of $\sin(nx)$.

5. Repeat the argument from #3 to show that for any $n = 1, 2, 3, \ldots$,

$$
\begin{aligned}
A_n &= \frac{-3}{n^2}\int_0^{2\pi}\Big(2\sin(x)\cos^2(x) - \sin^3(x)\Big)\cdot\cos(nx)\ dx, \\
\text{and}\qquad B_n &= \frac{-3}{n^2}\int_0^{2\pi}\Big(2\sin(x)\cos^2(x) - \sin^3(x)\Big)\cdot\sin(nx)\ dx.
\end{aligned}
$$

**Solution:** We apply integration by parts. Let $h(x) = \sin^2(x)\cos(x)$, and suppose $g'(x) = \sin(nx)$. Thus, $h'(x) \underset{\#1}{=} 2\sin(x)\cos^2(x) - \sin^2(x)$ and $g(x) = \frac{-1}{n}\cos(nx)$, so that

$$
\begin{aligned}
&\int_0^{2\pi} \sin^2(x)\cos(x)\cdot\sin(nx)\ dx \\
&= \int_0^{2\pi} h(x)\cdot g'(x)\ dx = h(x)\cdot g(x)\Big|_{x=0}^{x=2\pi} - \int_0^{2\pi} h'(x)\cdot g(x)\ dx \\
&= \frac{-1}{n}\sin^2(x)\cos(x)\cdot\cos(nx)\Big|_{x=0}^{x=2\pi} + \frac{1}{n}\int_0^{2\pi}\Big(2\sin(x)\cos^2(x) - \sin^2(x)\Big)\cdot\cos(nx)\ dx \\
&= \frac{1}{n}\Big(\sin^2(2\pi)\cos(2\pi)\cdot\sin(2n\pi) - \sin^2(0)\cos(0)\cdot\sin(0)\Big) \\
&\qquad + \frac{1}{n}\int_0^{2\pi}\Big(2\sin(x)\cos^2(x) - \sin^2(x)\Big)\cdot\cos(nx)\ dx \\
&\underset{(P)}{=} \frac{1}{n}\int_0^{2\pi}\Big(2\sin(x)\cos^2(x) - \sin^2(x)\Big)\cdot\cos(nx)\ dx.
\end{aligned}
$$

where equality $(P)$ is because $\sin^2(2\pi)\cos(2\pi)\cdot\sin(2n\pi) = \sin^2(0)\cos(0)\cdot\sin(0)$. Thus,

$$
A_n \underset{(\text{by}\#3)}{=} \frac{-3}{n}\int_0^{2\pi}\sin(x)^2\cos(x)\cdot\sin(nx)\,dx = \frac{-3}{n^2}\int_0^{2\pi}\Big(2\sin(x)\cos^2(x) - \sin^3(x)\Big)\cdot\cos(nx)\,dx.
$$

The proof for $B_n$ is similar.

6. Repeat the argument from #4 to conclude that, for all $n = 1, 2, 3, \ldots$,

$$|A_n| \quad \leq \quad \frac{18\pi}{n^2} \quad \text{and} \quad |B_n| \quad \leq \quad \frac{18\pi}{n^2}.$$

(For example, $A_{60} < \frac{\pi}{200}$.) Hence, there is *very* little energy in the 'high frequency' vibrations. **Solution:** From #5, we know that $A_n = \frac{-1}{n^2} \int_0^{2\pi} f''(x) \cdot \cos(nx)\, dx$. From #2, we know that $|f''(x)| \leq 9$ for all $x \in [0, 2\pi]$. Thus,

$$\begin{aligned} |A_n| &= \left| \frac{-1}{n^2} \int_0^{2\pi} f''(x) \cdot \cos(nx)\, dx \right| = \frac{1}{n^2} \left| \int_0^{2\pi} f''(x) \cdot \cos(nx)\, dx \right| \\ &\leq_{(*)} \frac{1}{n^2} \int_0^{2\pi} \left| f''(x) \cdot \cos(nx) \right| \, dx = \frac{1}{n^2} \int_0^{2\pi} \left| f''(x) \right| \cdot |\cos(nx)| \, dx \\ &\leq \frac{1}{n^2} \int_0^{2\pi} 9 \cdot 1\, dx = \frac{1}{n^2} 18\pi = \frac{18\pi}{n^2}. \end{aligned}$$

Here, inequality $(*)$ is by `Comparison Property` #8 on page 387 of §5.2.

The argument for $B_n$ is the same, only with $\sin(nx)$ instead of $\cos(nx)$.

7. A function $f : [0, 2\pi] \longrightarrow \mathbb{R}$ is called **smoothly periodic** if:

- $f(2\pi) = f(0)$;
- $f'(2\pi) = f'(0)$;
- $f''(2\pi) = f''(0)$;
- .....and, for all $k$, $f^{(k)}(2\pi) = f^{(k)}(0)$, where $f^{(k)}$ is the $k$th derivative of $f(x)$.

(For example, $f(x) = \sin^2(x)$ is smoothly periodic. )

Generalize the previous argument: Show that, if $f$ is *any* smoothly periodic function, then for any $k = 1, 2, 3, \ldots$,

$$A_n = \frac{\pm 1}{n^k} \int_0^{2\pi} f^{(k)}(x) \cdot \mathbf{C}_k(nx)\, dx, \quad \text{and} \quad B_n = \frac{\pm 1}{n^k} \int_0^{2\pi} f^{(k)}(x) \cdot \mathbf{S}_k(nx)\, dx.$$

Here, if $k$ is *even*, then we define $\mathbf{C}_k(x) = \cos(x)$ and $\mathbf{S}_k(x) = \sin(x)$; on the other hand, if $k$ is *odd*, then we define $\mathbf{C}_k(x) = \sin(x)$ and $\mathbf{S}_k(x) = \cos(x)$.

(**Hint:** *Proceed by induction on* $k$)

Conclude that, for any $k = 1, 2, 3, \ldots$ there is some constant $\ell_k$ such that

$$|A_n| \quad \leq \quad \frac{2\pi\ell_k}{n^k} \quad \text{and} \quad |B_n| \quad \leq \quad \frac{2\pi\ell_k}{n^k}.$$

Give a physical interpretation of this result.

**Solution:** We prove the result by induction on $k$. Question #3 showed it's true for $k = 1$ and question #5 showed it when $k = 2$. Suppose it is true for $k$; we want to prove it for $(k+1)$.

We apply integration by parts. Let $h(x) = f^{(k)}$ and suppose $g'(x) = \mathbf{C}_k(nx)$. Then $h'(x) = f^{(k+1)}(x)$ and $g(x) = \frac{\pm 1}{n}\mathbf{C}_{k+1}(nx)$. Thus,

$$\begin{aligned}
A_n &\underset{\text{(H)}}{=} \frac{\pm 1}{n^k}\int_0^{2\pi} f^{(k)}(x)\cdot \mathbf{C}_k(nx)\,dx \quad = \quad \frac{\pm 1}{n^k}\int_0^{2\pi} h(x)\cdot g'(x)\,dx \\
&= \frac{\pm 1}{n^k}\left( h(x)\cdot g(x)\,\Big|_{x=0}^{x=2\pi} \mp \int_0^{2\pi} h'(x)\cdot g(x)\,dx \right) \\
&= \frac{\pm 1}{n^k}\left( \frac{1}{n} f^{(k)}(x)\mathbf{C}_{k+1}(nx)\,\Big|_{x=0}^{x=2\pi} \mp \frac{1}{n^{k+1}}\int_0^{2\pi} f^{(k+1)}(x)\mathbf{C}_{k+1}(nx)\,dx \right) \\
&= \frac{\pm 1}{n^{k+1}}\left( f^{(k)}(2\pi)\mathbf{C}_{k+1}(2n\pi) - f^{(k)}(0)\mathbf{C}_{k+1}(0) \mp \int_0^{2\pi} f^{(k+1)}\mathbf{C}_{k+1}(nx)\,dx \right) \\
&\underset{\text{(P)}}{=} \frac{\pm 1}{n^{k+1}}\int_0^{2\pi} f^{(k+1)}(x)\mathbf{C}_{k+1}(nx)\,dx
\end{aligned}$$

Here, $(H)$ is by induction hypothesis, and $(P)$ is because $f$ is smoothly periodic, so that

$$f^{(k)}(2\pi)\mathbf{C}_{k+1}(2n\pi) \quad = \quad f^{(k)}(0)\mathbf{C}_{k+1}(0).$$

Now, since $f^{(k+1)}$ and $\mathbf{C}_{k+1}$ are continuous, there is some constant $\ell_{k+1} > 0$ so that $\left| f^{(k+1)}(x)\mathbf{C}_{k+1}(nx)\right| \le \ell_{k+1}$ for all $x \in [0, 2\pi]$. Thus,

$$\begin{aligned}
|A_n| &= \left| \frac{\pm 1}{n^{k+1}}\int_0^{2\pi} f^{(k+1)}(x)\mathbf{C}_{k+1}(nx)\,dx \right| \quad = \quad \frac{1}{n^{k+1}}\left| \int_0^{2\pi} f^{(k+1)}(x)\mathbf{C}_{k+1}(nx)\,dx \right| \\
&\le \frac{1}{n^{k+1}}\int_0^{2\pi} \left| f^{(k+1)}(x)\mathbf{C}_{k+1}(nx)\right| dx \quad \le \quad \frac{1}{n^{k+1}}\int_0^{2\pi} \ell_{k+1}\,dx \\
&= \frac{1}{n^{k+1}} 2\pi\ell_{k+1}.
\end{aligned}$$

The argument for $B_n$ is the same; just exchange the roles of $\mathbf{C}_k$ and $\mathbf{S}_k$.

**Physical interpretation:** If $f(x)$ is a smoothly periodic function, then the Fourier coefficients of $f(x)$ become small faster than $\frac{1}{n^{k+1}}$, for *any* choice of $k$. In other words, they get small very, *very* quickly as $n \to \infty$. This means that the 'high frequency' component of $f(x)$ contains very little energy.