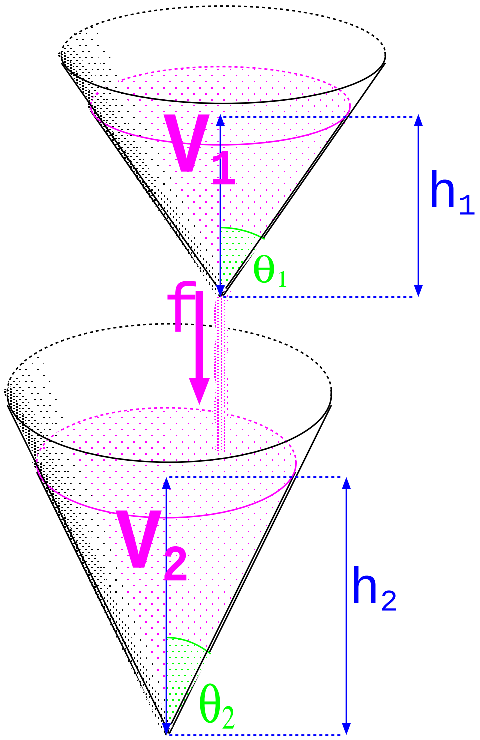
Math 110 —Assignment #5

Due: Monday December 2

- *Justify your answers.* Show all steps in your computations.
- Please indicate your final answer by putting a $\boxed{\text{box}}$ around it.
- Please write neatly and legibly. *Illegible answers will not be graded.*
- **Math 110A:** When finished, please give your assignment to Stefan or leave it under his door.
- **Math 110B:** When finished, please place your assignment in slot marked MATH 110 in the big white box outside the Math Department Office in Lady Eaton College.



If $\mathbf{C}$ is a cone of angle θ and height h, recall from HW #3 that the volume of $\mathbf{C}$ is given by $V(h) = \frac{\pi}{3}\tan(\theta)^2 h^3$. Use this information to answer the following questions:

In a winery, there are two tanks, $\mathbf{C}_1$ and $\mathbf{C}_2$, in the shape of inverted cones. Tank $\mathbf{C}_1$ has angle $\theta_1 = \frac{\pi}{3}$, and is initially full of wine to a depth of 10 metres. Tank $\mathbf{C}_2$ has angle $\theta_2 = \frac{\pi}{6}$, and is initially empty. Tank $\mathbf{C}_1$ drains into $\mathbf{C}_2$. The wine level in $\mathbf{C}_1$ drops steadily at a constant rate of 1 metre per second until it is empty after 10 seconds.

1. Let $h_1(t)$ be the wine level in $\mathbf{C}_1$ at time t. Thus $h_1(0) = 10$ and $h_1(10) = 0$. Find an expression for $h_1(t)$, for $t \in [0, 10]$.

Solution: $h_1(t) = \boxed{10 - t}$.

2. Let $V_1(t)$ be the volume of wine in $\mathbf{C}_1$ at time t. Find an expression for $V_1(t)$, for $t \in [0, 10]$.

Solution: $V_1(t) = \frac{\pi}{3}\tan(\theta_1)^2 h_1(t)^3 = \frac{\pi}{3}\tan(\pi/3)^2 \cdot (10 - t)^3 = \frac{\pi}{3} \cdot (\sqrt{3})^2 \cdot (10 - t)^3 = \boxed{\pi(10 - t)^3}$.

3. Let $f(t)$ be the rate at which wine is flowing out of tank $\mathbf{C}_1$ at time t. Find an expression for $f(t)$ for $t \in [0, 10]$.

Solution: $V_1(t) = 10 - \int_0^t f(t)\, dt$. Thus, by the Fundamental Theorem of calculus, $f(t) = -V_1'(t) = \boxed{3\pi \cdot (10 - t)^2.}$

4. Let $h_2(t)$ be the wine level in $\mathbf{C}_2$ at time t. Thus $h_2(0) = 0$. Find an expression for $h_2(t)$, for $t \in [0, 10]$.

Solution: The initial volume of wine in $\mathbf{C}_1$ is $V = V_1(0) = \pi(10)^3 = 1000\pi$. Thus, at time t, the amount of wine which has left $\mathbf{C}_1$, and entered $\mathbf{C}_2$, is $V_2(t) = V - V_1(t) = 1000\pi - \pi(10 - t)^3 = \pi \cdot \left(1000 - (1000 - 300t + 30t^2 - t^3)\right) = \pi \cdot (t^3 - 30t^2 + 300t)$.

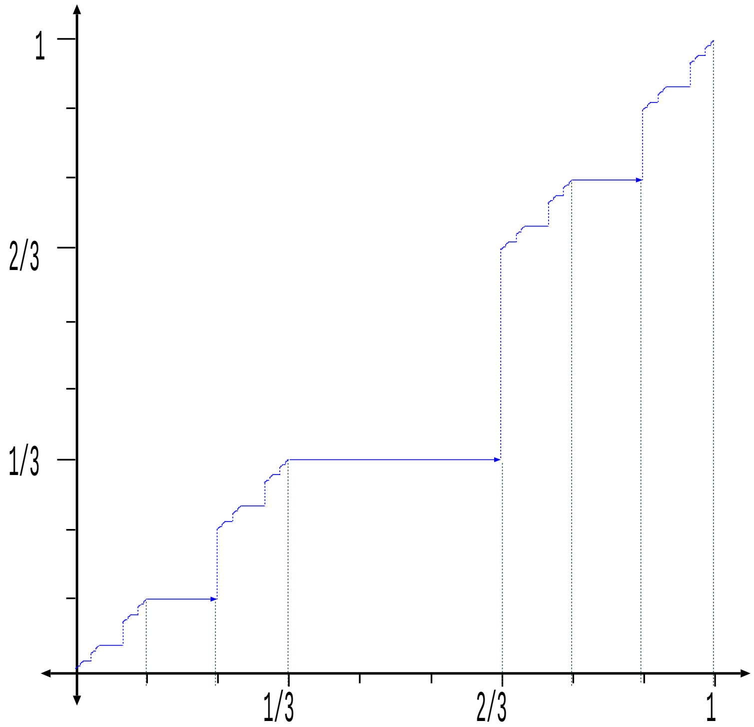


Figure 1: The Devil's staircase.

However, we also know that $V_2(t) = \frac{\pi}{3}\tan(\theta_2)^2 h_2(t)^3 = \frac{\pi}{3}\tan(\pi/6)^2 \cdot h_2(t)^3 = \frac{\pi}{3}\cdot(\frac{1}{\sqrt{3}})^2 \cdot h_2(t)^3 = \frac{\pi}{9}\cdot h_2(t)^3$.

Thus, $h_2(t) = \sqrt[3]{\frac{9}{\pi}V_2(t)} = \sqrt[3]{\frac{9}{\pi}\pi\cdot(t^3 - 30t^2 + 300t)} = \boxed{\sqrt[3]{9t\,(t^2 - 30t + 300)}}$.

Bonus Problem: Any real number $\alpha \in [0,1]$ has a unique[1] **trinary representation** $0.a_1a_2a_3a_4\ldots$ so that $\alpha = \sum_{n=0}^{\infty}\frac{a_n}{3^n}$. For example, in trinary notation,

$$\frac{1}{3} = 0.1;\quad \frac{2}{3} = 0.2;\quad \frac{1}{9} = 0.01;\quad \frac{1}{2} = 0.1111111\ldots;$$

A **Cantor number** is a number whose trinary expansion contains only 0's and 2's, and has no 1's. The **Devil's staircase** (see Figure 1) is the function $f:[0,1]\longrightarrow[0,1]$ so that $f(\alpha) = \beta$, where β is the largest Cantor number less than or equal to α. For example, $f(0.120) = 0.020$, $f(0.1000\ldots) = 0.0222\ldots$, and $f(0.122121000012) = 0.022020222202$.

Try to compute the integral $\int_0^1 f(x)\ dx$, as a limit of right-hand Riemann sums. Now do it using left-hand Riemann sums. Do your answers agree? If not, why not?

[1] Well, *almost* unique. If $\alpha = 0.a_1a_2a_3\ldots a_{n-1}a_n000\ldots$, then we could also write $\alpha = 0.a_1a_2a_3\ldots a_{n-1}b_n2222\ldots$, where $b_n = a_n - 1$. This is analogous to the fact that $0.19999\ldots = 0.2$ in decimal notation.