Math 110 —Assignment #3 —Solutions

Due: Monday, October 28th

1. **Chain Rule for Piecewise linear functions:** Let $h = f \circ g$, where

$$f(y) = \begin{cases} 2y+4 & \text{if } y \le -1 \\ -2y & \text{if } -1 \le y < 1 \\ 3y-5 & \text{if } 1 < y \end{cases} \qquad \text{(Figure A)}$$

$$\text{and} \quad g(x) = \begin{cases} 3x & \text{if } x \le 0 \\ x/2 & \text{if } 0 < x \end{cases} \qquad \text{(Figure B)}$$

(a) Express $h(x)$ as a **piecewise linear** function, similar to f and g. In other words, find real numbers $X_1 < X_2 < X_3$, slopes m_0, m_1, m_2, m_3 and values b_0, b_1, b_2, b_3 so that

$$h(x) = \begin{cases} m_0x+b_0 & \text{if } x \le X_1 \\ m_1x+b_1 & \text{if } X_1 < x \le X_2 \\ m_2x+b_2 & \text{if } X_2 < x \le X_3 \\ m_3x+b_3 & \text{if } X_3 < x \end{cases}.$$

Solution: $h(x) = \begin{cases} 6x+4 & \text{if } x \le \frac{-1}{3} \\ -6x & \text{if } \frac{-1}{3} < x \le 0 \\ -x & \text{if } 0 < x \le 2 \\ 3x/2-5 & \text{if } 2 < x \end{cases}$. To see this, observe:

- If $x \le \frac{-1}{3}$, then $x \le 0$, so $g(x) = 3x$. Thus, if $y = g(x)$, then $y < 3 \cdot \frac{-1}{3} = -1$. Thus, $f(y) = 2y+4 = 2\cdot g(x)+4 = 2\cdot 3x+4 = 6x+4$.
- If $\frac{-1}{3} < x \le 0$, then $x \le 0$, so $g(x) = 3x$. Thus, if $y = g(x)$, then $-1 = 3\cdot\frac{-1}{3} < y \le 3\cdot 0 = 0$. Thus, $f(y) = -2y = -2\cdot g(x) = -2\cdot 3x = -6x$.
- If $0 < x \le 2$, then $0 < x$, so $g(x) = \frac{x}{2}$. Thus, if $y = g(x)$, then $0 = \frac{0}{2} < y \le \frac{2}{2} = 1$. Thus, $f(y) = -2y = -2\cdot g(x) = -2\frac{x}{2} = -x$.
- If $2 < x$, then $0 < x$, so $g(x) = \frac{x}{2}$. Thus, if $y = g(x)$, then $1 = \frac{2}{2} < y$. Thus, $f(y) = 3y+5 = 3\cdot g(x) - 5 = 3\frac{x}{2} - 5$.

□

(b) Observe that h is differentiable on each of the intervals $(-\infty, X_1)$, (X_1, X_2), (X_2, X_3), and (X_3, ∞). Verify that the **Chain Rule** holds on each of these intervals.

Solution: h is differentiable on each of the domains $\left(-\infty, \frac{-1}{3}\right)$, $\left(\frac{-1}{3}, 0\right)$, $(0,2)$, and $(2,\infty)$, because it is *linear* on each of these domains. To see that the chain rule holds, observe that

- If $x \le \frac{-1}{3}$, then $h'(x) = 6$. But if $x \le \frac{-1}{3}$, then $g'(x) = 3$. If $y = g(x)$, then $y < -1$, so $f'(y) = 2$. Thus, $f'(y)\cdot g'(x) = 2\cdot 3 = 6 = h'(x)$.
- If $\frac{-1}{3} < x \le 0$, then $h'(x) = -6$. But if $\frac{-1}{3} < x \le 0$, then $g'(x) = 3$, and if $y = g(x)$, then $f'(y) = -2$. Thus, $f'(y)\cdot g'(x) = -2\cdot 3 = -6 = h'(x)$.
- If $0 < x \le 2$, then $h'(x) = -1$. But if $0 < x \le 2$, then $g'(x) = \frac{1}{2}$, and if $y = g(x)$, then $f'(y) = -2$. Thus, $f'(y)\cdot g'(x) = -2\cdot\frac{1}{2} = -1 = h'(x)$.
- If $2 < x$, then $h'(x) = \frac{3}{2}$. But if $2 < x$, then $g'(x) = \frac{1}{2}$, and if $y = g(x)$, then $f'(y) = 3$. Thus, $f'(y)\cdot g'(x) = 3\cdot\frac{1}{2} = \frac{3}{2} = h'(x)$.

□

2. If $\mathbf{C}$ is a cone of height h and base radius r (Figure C), recall that the volume of $\mathbf{C}$ is $\frac{\pi}{2}r^2h$.

The *angle of repose* of a granular material (eg. sand, grain, etc.) is the steepest angle at which the material can be sloped without avalanching. If sand is poured from a high place onto a single spot on the ground, it will form a cone whose angle θ is the angle of repose.

(a) Suppose $\theta = \frac{\pi}{6}$ is the angle of repose. Compute the radius of a sand cone of height h. Now compute the volume.

Solution: Note that $\tan(\theta) = \frac{h}{r}$. Thus, $r = h/\tan(\theta) = h \cdot \cot\left(\frac{\pi}{6}\right) \;=\; \boxed{\sqrt{3} \cdot h.}$

Thus, the volume is $v(h) = \frac{\pi}{2} r^2 h \;=\; \frac{\pi}{2}\left(\sqrt{3} \cdot h\right)^2 \cdot h \;=\; \boxed{\frac{3\pi}{2} h^3.}$ □

(b) Suppose sand is being poured onto the cone at a constant rate of 10 $\mathtt{m}^3/\sec$. After some time, the sand cone is 5 metres high. At this instant, how fast is the cone's height increasing, in metres per second?

Solution: Let $V(t)$ be the volume of sand in the cone at time t. Since sand is being added to the cone at a constant rate of 10 $\mathtt{m}^3/\sec$, we know that $V'(t) = 10$. (In other words, V is a linear function with slope 10.)

Let $h(t)$ be the height of the pile at time t. Then we know from part (a) that $V(t) \;=\; v\Big(h(t)\Big)$, where $v(h) \;=\; \frac{3\pi}{2} h^3$. Thus, applying the **Chain rule**, we have:

$$V'(t) \quad = \quad v'\Big(h(t)\Big) \cdot h'(t) \tag{1}$$

Since $v(h) \;=\; \frac{3\pi}{2} h^3$, it follows that $v'(h) \;=\; \frac{9\pi}{2} h^2$. Also, we've already established that $V'(t) = 10$. Substituting these expressions into (1), we obtain:

$$10 \quad = \quad \frac{9\pi}{2}\Big(h(t)\Big)^2 \cdot h'(t) \tag{2}$$

If t is the instant when the cone is 5 metres high, then $h(t) = 5$. Substitute into (2) to get:

$$10 \quad = \quad \frac{9\pi}{2}(5)^2 \cdot h'(t) \quad = \quad \frac{9\pi}{2} \cdot 25 \cdot h'(t) \quad = \quad \frac{225\pi}{2} \cdot h'(t)$$

and conclude that $h'(t) \;=\; \frac{20}{225\pi} \;=\; \boxed{\frac{4}{45\pi}}$. □

3. **Bonus problem:** Prove that $\cos'(x) = -\sin(x)$, using the definition: $f'(x) \;=\; \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

Solution: We use the following facts:

(a) $\cos(x+h) \;=\; \cos(x)\cos(h) - \sin(x)\sin(h)$.

(b) $\displaystyle\lim_{h \to 0} \frac{\cos(h) - 1}{h} \;=\; 0$.

(c) $\displaystyle\lim_{h \to 0} \frac{\sin(h)}{h} \;=\; 1$.

(a) is a standard trigonometric identity; **(b)** and **(c)** were established in class.

$$\begin{aligned}
\cos'(x) &= \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h} \underset{\text{by (a)}}{=\!=\!=} \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \cos(x)}{h} - \lim_{h \to 0} \frac{\sin(x)\sin(h)}{h} \\
&= \cos(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h \to 0} \frac{\sin(h)}{h} \underset{\text{(b)\&(c)}}{=\!=\!=} \cos(x) \cdot 0 - \sin(x) \cdot 1 \quad = \quad -\sin(x)
\end{aligned}$$

□