Math 110 —Assignment #2

Due: Monday, October 7th

1. An archer takes aim at a target. If he releases the arrow at an angle of θ to the horizontal, then it will strike the target at a distance (in millimetres) of $f(\theta)$ from the bull's eye, where

$$f(\theta) \quad = \quad \sin(\theta) \cdot (\theta^2 - 900).$$

Thus, for example, if he releases the arrow at an angle of precisely 30^o to the horizontal, he will score a bull's eye, whereas if he releases the arrow at 45^o, he will miss the bull's eye by a distance of $\sin(45^o) \cdot (45^2 - 900) \; = \; \frac{1}{\sqrt{2}} 1125 \text{ mm} \; \approx \; 795 \text{ mm}$, or 79.5 cm.

The archer's aim is good enough that he always releases the arrow at an angle of between 20^o and 40^o.

(3 pts) (a) How close to 30^o must he release the arrow to hit within 1 mm of the target? (**Hint:** *Find some δ so that, if $|\theta - 30| < \delta$, then $\left|f(\theta)\right| < 1$. It doesn't have to be the best possible δ*).

Solution: We want to find a $\delta > 0$ so that, if $|x - 30| < \delta$, then $|f(x) < 1$. First, write an expression for $|f(x)|$:

$$\begin{aligned} |f(\theta)| \quad &= \quad \left|\sin(\theta) \cdot (\theta^2 - 900)\right| \quad = \quad |\sin(\theta)| \cdot |(\theta + 30) \cdot (\theta - 30)| \\ &= \quad |\sin(\theta)| \cdot |\theta + 30| \cdot |\theta - 30|. \end{aligned} \tag{1}$$

Since $-1 \; \leq \; \sin(\theta) \; \leq \; 1$, we know that $|\sin(\theta)| \; \leq \; 1$. Thus, we can simplify expression (1) to:

$$|f(\theta)| \quad = \quad |\sin(\theta)| \cdot |\theta + 30| \cdot |\theta - 30| \quad \leq \quad 1 \cdot |\theta + 30| \cdot |\theta - 30| \quad = \quad |\theta + 30| \cdot |\theta - 30| \tag{2}$$

Next, recall that $20 \; < \; \theta \; < \; 40$. Thus, adding 30 to everything, we get: $50 \; = \; 20 + 30 \; < \; \theta + 30 \; < \; 40 + 30 \; = \; 70$. Since $0 \; < \; \theta + 30 \; < \; 70$, we conclude that $|\theta + 30| < 70$. Thus, we can simplify (2) to:

$$|f(\theta)| \quad \leq \quad |\theta + 30| \cdot |\theta - 30| \quad \leq \quad 70 \cdot |\theta - 30| \tag{3}$$

So, if $|\theta - 30| \; < \; \delta$, then (3) implies that $|f(\theta)| \; < \; 70\delta$. So, if $\delta = \frac{1}{70}$, then $|f(\theta)| \; < \; 70\delta \; < \; 1$.

Conclusion: If the archer releases the arrow within the range of $30 - \frac{1}{70}$ to $30 + \frac{1}{70}$, then the arrow will strike within 1 mm of the bull's eye. ☐

(2 pts) (b) Generalize this argument to show that $\lim_{\theta \to 30} f(\theta) = 0$. Conclude that f is continuous at $\theta = 30$.

Solution: For any $\epsilon > 0$, we need a $\delta > 0$ so that, if $0 < |x - 30| < \delta$, then $|f(\theta)| < \epsilon$. In part (a), we showed:

$$\text{If } |\theta - 30| \; < \; \delta, \text{ then } |f(\theta)| \; < \; 70\delta. \tag{4}$$

So, given $\epsilon > 0$, let $\delta = \frac{\epsilon}{70}$. Then $70\delta \; = \; 70\frac{\epsilon}{70} \; = \; \epsilon$. Hence, if $|\theta - 30| \; < \; \delta$, then (4) implies that $|f(\theta)| \; < \; 70\delta = \epsilon$, as desired.

Thus, $\lim_{\theta \to 30} f(\theta) = 0$. But $f(30) = 0$, hence $\lim_{\theta \to 30} f(\theta) \; = \; f(30)$, so f is continuous at $\theta = 30$. ☐

(5 pts) 2. Let $f(x) = \dfrac{\sin(x)(x^2+1)}{x^3-5}$.

Use an $\epsilon - N$ proof to show that $\lim_{x\to\infty} f(x) = 0$. That is: for any $\epsilon > 0$, you must find some $N > 0$ so that, if $x > N$, then $|f(x)| < \epsilon$.

Solution: Divide numerator and denominator by x^3 to get:

$$f(x) = \frac{x^{-3}}{x^{-3}} \cdot \frac{\sin(x)(x^2+1)}{x^3-5} = \frac{\sin(x)\left(\frac{1}{x}+\frac{1}{x^3}\right)}{1-\frac{5}{x^3}}$$

Thus

$$\left|f(x)\right| = \frac{|\sin(x)| \cdot \left|\frac{1}{x}+\frac{1}{x^3}\right|}{\left|1-\frac{5}{x^3}\right|} \le \frac{\left|\frac{1}{x}+\frac{1}{x^3}\right|}{\left|1-\frac{5}{x^3}\right|} \tag{5}$$

Now, if $x > 2$, then $x^3 > 8$, so $\frac{3}{8} = 1-\frac{5}{8} < 1-\frac{5}{x^3}$, so

$$\frac{3}{8} < \left|1-\frac{5}{x^3}\right|$$

If we take reciprocals, we must reverse the direction of the inequalities, to get:

$$\frac{1}{\left|1-\frac{5}{x^3}\right|} < \frac{8}{3}$$

so (5) simplifies to

$$\left|f(x)\right| \le \frac{\left|\frac{1}{x}+\frac{1}{x^3}\right|}{\left|1-\frac{5}{x^3}\right|} \le \frac{8}{3}\cdot\left|\frac{1}{x}+\frac{1}{x^3}\right| = \frac{8}{3}\cdot\left|\frac{1}{x}\right|\cdot\left|1+\frac{1}{x^2}\right| \tag{6}$$

Now, if $x > 2$, then $x^2 > 4$, so $0 < \frac{1}{x^2} < \frac{1}{4}$, and so $0 < 1+\frac{1}{x^2} < 1+\frac{1}{4} = \frac{5}{4}$. Hence, $\left|1+\frac{1}{x^2}\right| < \frac{5}{4}$. Hence, (6) simplifies to:

$$\left|f(x)\right| \le \frac{8}{3}\cdot\left|\frac{1}{x}\right|\cdot\left|1+\frac{1}{x^2}\right| \le \frac{8}{3}\cdot\frac{5}{4}\cdot\left|\frac{1}{x}\right| = \frac{10}{3}\left|\frac{1}{x}\right| \tag{7}$$

Now, we want $\left|f(x)\right| < \epsilon$. From (7), we know it is sufficient to have

$$\frac{10}{3}\left|\frac{1}{x}\right| < \epsilon \tag{8}$$

which is equivalent to

$$\left|\frac{1}{x}\right| < \frac{3\epsilon}{10} \tag{9}$$

which is equivalent to

$$|x| > \frac{10}{3\epsilon} \tag{10}$$

Working backwards, we conclude: if $x > 2$, *and* $x > \dfrac{10}{3\epsilon}$, then (10) is satisfied, which is equivalent to (9), which is equivalent to (8), which implies $\left|f(x)\right| < \epsilon$, which is what we wanted. ______________ $\square$

Bonus Problem: *(For fun)*

Recall that the **integers** are the numbers $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$.

A real number x is **rational** if $x = \frac{n}{m}$ for some integers n and m. For example, -2, 0, and $\frac{4}{3}$ are rational.

A real number x is **irrational** if it is not rational. For example, π and $\sqrt{2}$ are irrational.

1. Show any number with a finite decimal expansion (eg. 0.1342) is rational. (**Hint:** $0.1342 = \frac{1342}{10000}$)

2. Conclude that the rational numbers are *dense*, meaning that, for any real numbers x, and any $\epsilon > 0$ there is a rational number q with $|x - q| < \epsilon$. (**Hint:** *Consider the decimal expansion of x; truncate it and apply (1)*)

3. If r is rational, but i is irrational, then show that $r \cdot i$ is also irrational. (**Hint:** *Observe that $i = \frac{1}{r} r \cdot i$.*)

4. Conclude that that the irrational numbers are *dense*, meaning that, for any real numbers x, and any $\epsilon > 0$ there is an irrational number i with $|x - i| < \epsilon$. You may use the fact that $\sqrt{2}$ is irrational. (**Hint:** *Use (2) to find a rational number close to $x/\sqrt{2}$. Now multiply by $\sqrt{2}$ and apply (3)*)

5. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ as follows: $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

   Show that f is discontinuous at every irrational x. (**Hint:** *Apply (2)*)

6. Show that f is discontinuous at every nonzero rational x. (**Hint:** *Apply (4)*)

7. Show that f is continuous at 0. (**Hint:** *If y is close to 0, then either y is rational or irrational; either way show that $f(y)$ is close to $f(0) = 0$.*)

8. Conclude that f is discontinuous everywhere except at zero.