Trent University, Winter 2018

MATH–CCTH 1080H Test #2

~~Tuesday, 13~~ Friday, 16 March

Time: 60 minutes

Name: Solutions to Practice Test #2

Student Number: 0123456

Question	Mark	
1	___	
2	___	
3	___	
Total	___	/30

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. A card is drawn at random from a standard deck and replaced before the experiment is repeated. What is the expected number of $\diamondsuit$s in five such draws?

b. A fair standard die is rolled once and the random variable Y gives the number of the face that came up. Compute the expected value $E(Y)$ and variance $V(Y)$ of Y.

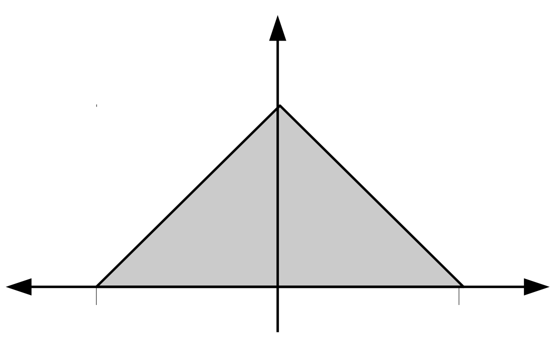
c. The continuous random variable X has $f(x) = \begin{cases} 1+x & -1 \leq x \leq 0 \\ 1-x & 0 \leq x \leq 1 \\ 0 & x < -1 \text{ or } x > 1 \end{cases}$ as its probability density function. What is the expected value of X?

SOLUTIONS. **a.** There are 13 $\diamondsuit$s out of 52 cards in a standard deck, so there is a $\frac{13}{52} = \frac{1}{4} = 0.25$ chance of getting a $\diamondsuit$ when randomly drawing a single card from the deck. Suppose the random variables X_1, X_2, X_3, X_4, and X_5 count the number of $\diamondsuit$s obtained on each of the five draws, respectively. Note that the X_i are independent and identically distributed, that $E(X_i) = 0 \cdot \frac{3}{4} + 1 \cdot \frac{1}{4} = \frac{1}{4}$ for each i, and that $X = X_1 + X_2 + X_3 + X_4 + X_5$ counts the total number of $\diamondsuit$s that occur in the five draws. The expected value of X is therefore:

$$\begin{aligned} E(X) &= E(X_1 + X_2 + X_3 + X_4 + X_5) \\ &= E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5) \\ &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{5}{4} = 1.25 \quad \square \end{aligned}$$

b. Since the die is standard, it has six faces that are numbered 1–6; since it is also fair, each face has a probability of $\frac{1}{6}$ of coming up. It follows that $E(Y) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = \frac{7}{2} = 3.5$. To compute $V(Y)$, we will also need $E(Y^2) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6} = \frac{91}{6}$. Then $V(Y) = E(Y^2) - [E(Y)]^2 = \frac{91}{6} - \left[\frac{7}{2}\right]^2 = \frac{91}{6} - \frac{49}{4} = \frac{182}{12} - \frac{147}{12} = \frac{35}{12}$. $\square$

c. The density function has a graph that is symmetric around the y-axis, also known as the line $x = 0$:



It follows that the expected value of X is $E(X) = 0$. $\blacksquare$

2. Do *all three (3)* of **a**–**c**.

Consider the following data: 2, 1, 5, 2, 6, 3

a. Find the mean, median, and mode of the given data. *[3]*

b. Find the standard deviation of the given data. *[4]*

c. Suppose the data gives the faces that came up in six rolls of a die. Can you determine whether the die is likely to be biased? Explain why or why not as best you can. *[3]*

SOLUTIONS. **a.** Note that we have six data points.

The mean is just the average of the data: $\mu = \frac{2+1+5+2+6+3}{6} = \frac{19}{6} = 3.1\dot{6}$.

For the median, we put the data in increasing order: 1, 2, 2, 3, 5, 6. Since there is an even number of data points, the median is the average of the middle two data points: $\frac{2+3}{2} = \frac{5}{2} = 2.5$.

The mode is the most common number among the data. In this case, the mode is 2: it occurs twice in the data and every other number in the data occurs only once. □

b. To compute the standard deviation of the given data, we first compute the variance, which is the average of the squares of the data minus the square of the mean of the data:

$$\begin{aligned}\sigma^2 &= \frac{2^2+1^2+5^2+2^2+6^2+3^2}{6} - \left(\frac{19}{6}\right)^2 = \frac{4+1+25+4+36+9}{6} - \frac{361}{36} \\ &= \frac{79}{6} - \frac{361}{36} = \frac{474}{36} - \frac{361}{36} = \frac{113}{36} = 3.13\dot{8} \approx 3.14\end{aligned}$$

The standard deviation is the (positive) square root of the variance: $\sigma = \sqrt{\frac{113}{36}} \approx 1.77$. □

c. We don't really have enough data to determine whether the die is biased. The data suggest that it *might* be biased in favour of the lower-numbered faces, since four out of six data points fall into the lower half of the range, but one would have to run a much larger experiment to be reasonably confifent of that conclusion. ■

3. Do *one (1)* of **a** or **b**. *[10]*

a. Rent University is a federation of three colleges, Complain College, Growly College, and Ought-Not-To-Be College. Each college pays rent to the university [hence the name!] for the use of the university's facilities at a rate of 4000 per student. The colleges, however, charge different amounts for tuition and what they actually get from each student varies due to scholarships, bursaries, whether students are from another province, and so on. The colleges' overall situations are as follows:

College	*Students*	*Total Income*
Complain	120	$450,000$
Growly	100	$410,000$
Ought-Not-To-Be	80	$400,000$

What is the average income per student for each college and for Rent University as whole? Determine whether each college and the university as a whole are making or losing money, and just how much.

b. You go to Tim Horton's to get two coffees at 3 a.m. because you need to study hard for your math test. The terribly bored staff prepare three sets of two coffees for you and tell you that one set has two Roll Up The Rim winners, one has a winner and a loser, and one has two losers, but give no indication which is which. You pick one of the sets of two coffees at random. After finishing your first coffee you check the rim and discover that you have a winner. What is the probability that your second coffee is also a winner?

SOLUTIONS. **a.** Each college's average income per student is just the ratio of income to the number of students for that college:

College	*Students*	*Total Income*	*Income per Student*
Complain	120	$450,000$	3750
Growly	100	$410,000$	4100
Ought-Not-To-Be	80	$400,000$	5000

Rent University, as noted in the set up, gets 4000 per student; the three colleges as a whole receive an average income per student of $\frac{450{,}000+410{,}000+400{,}000}{120+100+80} = \frac{1{,}260{,}000}{300} = 4200$.

Looking at all these numbers, Complain College loses $250 = 4000 - 3750$ per student (with a total loss of $30,000$), Growly College makes $100 = 4100 - 4000$ per student (with a total profit of $10,000$, and Ought-Not-To-Be College makes 1000 per student (with a total profit of $80,000$). The colleges as a whole make $200 = 4200 - 4000$ per student (with total profit of $60,000$. Whether rent University *per se* is making or losing money is impossible to determine, since we do not know the expenses it has. □

c. The probability that your second cup of coffee is a winner is $\frac{2}{3}$. Look up the Wikipedia article on the *Bertrand's Box Paradox* for an explanation ... ■

[Total = 30]