Trent University, Winter 2018

MATH–CCTH 1080H Test #1

~~Tuesday, 6~~ Friday, 9 February

Time: 60 minutes

Name: Practice Test Solutions

Student Number: 0123456

Question	Mark	
1	____	
2	____	
3	____	
Total	____	/30

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. Fill in the blanks: *i.* 5 is to 4 as __ is to 52. *ii.* __ is to 2.1 as 6 is to 9.
iii. 3 is to __ as 48 is to 40. *iv.* 19 is to 4.75 as 2.2 is to __.
v. 81 is to __ as __ is to 1. [Same number in both blanks!]

b. If a *zap* is 17 *tingles* and a *tingle* is 13 *shivers*, how many *zaps* are there in 41.99 *shivers*?

c. Suppose a fair coin is tossed until it comes up heads. What is the probability that no less than three and no more than six tosses will be required?

SOLUTIONS. **a.** *i.* $\frac{5}{4} = \frac{\_\_}{52}$ so $\_\_ = \frac{5 \times 52}{4} = \frac{260}{4} = 65$.

ii. $\frac{\_\_}{2.1} = \frac{6}{9}$ so $\_\_ = \frac{2.1 \times 6}{9} = \frac{12.6}{9} = 1.4$.

iii. $\frac{3}{\_\_} = \frac{48}{40}$ so $\frac{\_\_}{3} = \frac{40}{48}$ so $\_\_ = \frac{3 \times 40}{48} = \frac{120}{48} = 2.5$.

iv. $\frac{19}{4.75} = \frac{2.2}{\_\_}$ so $\frac{4.75}{19} = \frac{\_\_}{2.2}$ so $\_\_ = \frac{2.2 \times 4.75}{19} = 0.55$.

v. $\frac{81}{\_\_} = \frac{\_\_}{1}$ so $\_\_ \times \_\_ = 81 \times 1$ so $(\_\_)^2 = 81$ so $\_\_ = \sqrt{81} = 9$. □

b. One *zap* has 17 *tingles* and each *tingle* has 13 *shivers*, so 1 *zap* = 17 × 13 *shivers* = 221 *shivers*. It follows that 1 *shiver* = $\frac{1}{221}$ *zaps*, so 41.99 *shivers* = 41.99 × $\frac{1}{221}$ *zaps* = 0.19 *zaps*. □

c. Divide and conquer! Here goes:

$$\begin{aligned} P(3 \leq \text{tosses} \leq 6) &= P(TTH) + P(TTTH) + P(TTTTH) + P(TTTTTH) \\ &= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} \\ &= \frac{8}{64} + \frac{4}{64} + \frac{2}{64} + \frac{1}{64} = \frac{8+4+2+1}{64} = \frac{15}{64} = 0.234375 \end{aligned}$$ ■

2. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. If the speed of light in a vacuum is $300,000,000$ metres per second, what is it in kilometres per hour?

b. How many ways that can be told apart are there to arrange the letters in the word "unusual" if copies of the same letter can't be told apart?

c. A fair coin is tossed four times. What is the probability that at least two heads will come up?

SOLUTIONS. **a.** There are 1000 metres in a kilometre and $60 \times 60 = 3600$ seconds in an hour. It follows that:

$$\begin{aligned} 300,000,000\ m/s &= 300,000\ km/s = 300,000\ km/s \times 3600\ s/hr \\ &= 1,080,000,000\ km/hr \quad \square \end{aligned}$$

b. The word 'unusual" has seven letters, counting three copies of "u". If these were all distinguishable, *i.e.* could be told apart, there would be $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7! = 5040$ ways to arrange them. Since the three copies of "u" cannot be told apart, any of the $3 \cdot 2 \cdot 1 = 3! = 6$ ways they could be rearranged among themselves in any arrangement of the seven letters cannot be told apart. It follows that there are $\frac{7!}{3!} = \frac{5040}{6} = 840$ ways to arrange the letters that can be told apart. $\square$

c. Here goes:

$$\begin{aligned} P(\geq 2 \text{ heads}) &= 1 - P(< 2 \text{ heads}) \\ &= 1 - [P(0 \text{ heads}) + P(1 \text{ head})] \\ &= 1 - [P(TTTT) + P(HTTT) + P(THTT) + P(TTHT) + P(TTTH)] \\ &= 1 - \left[\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4\right] \\ &= 1 - \left[\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right] = 1 - \frac{5}{16} = \frac{11}{16} = 0.6875 \quad \blacksquare \end{aligned}$$

3. Do *one (1)* of **a** or **b**. *[10]*

a. One of our few sources for the life of the mathematician Diophantus (*c.* 250 A.D.) is a puzzle given in an anthology by a Fifth Century writer named Metrodorus:

> 'Here lies Diophantus,' the wonder behold.
> Through art algebraic, the stone tells how old:
> 'God gave him his boyhood one-sixth of his life,
> One twelfth more as youth while whiskers grew rife;
> And then yet one-seventh ere marriage begun;
> In five years there came a bouncing new son.
> Alas, the dear child of master and sage
> After attaining half the measure of his father's life
> chill fate took him.
> After consoling his fate by the science of numbers
> for four years, he ended his life.'

According to the riddle, how old was Diophantus when he died?

b. A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet two inhabitants: Zed and Zoey. Zed says that it's false that Zoey is a knave. Zoey claims, "Zed and I are different." Determine who is a knight and who is a knave. Please give your reasoning.

SOLUTIONS. **a.** This one is probably best done with a bit of algebra. Suppose x is Diophantus' age when he died. We are told that $x = \frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4$. [Well, that's the accepted interpretation of the riddle ...] The least common multiple of the denominators (6, 12, 7, and 2) of the fractions that appear is 84, so we'll first put the fractions over a common denominator of 84 and then consolidate:

$$\begin{aligned} x &= \frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4 = \frac{14}{84}x + \frac{7}{84}x + \frac{12}{84} + \frac{42}{84}x + 9 \\ &= \frac{14+7+12+42}{84}x + 9 = \frac{75}{84}x + 9 \end{aligned}$$

It follows that $x - \frac{75}{84}x = \frac{9}{84}x = 9$, so $x = \frac{84 \times 9}{9} = 84$. $\square$

b. Suppose Zed were a knight. Then his assertion that it is false that Zoey is a knave would mean that Zoey is also a knight. This, in turn, would mean that Zoey's statement that Zed and she are of different kinds was false, which would contradict the conclusion that she was a knight. The only way out of this is for Zed to be a knave.

Since Zed must be a knave, his statement that it is false that Zoey is a knave must itself be false, so Zoey must be a knave. There is no contradiction this time, because if Zed and Zoey are both knaves, her statement that they are of different kinds is false, as a knave's statement should be.

Thus Zed and Zoey are both knaves. ■

[Total = 30]