MATH-CCTH 1080H – Mathematics for Everyday Life
TRENT UNIVERSITY, Winter 2018 in Peterborough

Solution to Assignment #5
Faking fairness

1. You are stuck on a desert island with a friend and single coin which is your only source of randomness. [Deterministic desert islands are very boring!] The coin in question is biased in that it comes up heads more often than tails, though you do not know the exact bias. (It does come up tails some of the time.) You and your friend want to play a game that needs a fair coin. How could you use the biased coin to simulate the fair coin that you need? *[10]*

SOLUTION. The coin with unknown bias can be used to simulate a fair coin as follows:

1. Toss the biased coin twice.
2. If the outcome of the double toss is HT, return H for the toss of the fair coin, if the outcome of the double toss is TH, return T for the toss of the fair coin, and if the outcome of the double toss is HH or TT, go back to step one and repeat the experiment.

Why does this work?

First, whatever the bias of the given coin may be, the two tosses of this coin are independent of each other, so $P(HT) = P(H)\cdot P(T) = P(T)\cdot P(H) = P(TH)$. This means that the experiment has an equal probability of returning a H as it does returning a T for the simulated fair coin, as desired.

Second, the probability that the process will repeat at least n times is $[P(HH) + P(TT)]^n$, and because $P(HH) + P(TT) < 1$ for the biased coin, it follows that the probability it will repeat forever is 0. (The probability that the process will repeat n times is $[P(HH) + P(TT)]^n$; since $P(HH) + P(TT) < 1$, this probability gets arbitrarily close to 0 as n increases.) Thus the given process will eventually generate an outcome for the fair coin being simulated. □