MATH-CCTH 1080H – Mathematics for Everyday Life
TRENT UNIVERSITY, Winter 2018 in Peterborough

Solution to Assignment #4
Toss, toss, toss a coin ...

1. A fair coin has an equal probability of coming up heads or tails. Such a coin is tossed until the second time a head comes up. What is the probability that this will require more than five tosses? *[10]*

SOLUTION. We'll take a semi-brute force approach: $P(> 5 \text{ tosses}) = 1 - P(\leq 5 \text{ tosses})$, and the probability that the experiment will require at most five tosses can be computed by brute force fairly readily, if tediously.

The sequences of no more than five tosses where the second head comes up on the last toss are HH, HTH, TTH, $HTTH$, $THTH$, $TTHH$, $HTTTH$, $THTTH$, $TTHTH$, and $TTTHH$. Since we are tossing a fair coin, each of these sequences has a probability of $\left(\frac{1}{2}\right)^n$, where n is the number of tosses in the sequence. Thus the probability that the experiment will require at most five tosses is

$$\begin{aligned}
P(\leq 5 \text{ tosses}) &= P(HH) + P(HTH) + P(THH) + P(HTTH) + P(THTH) + P(TTHH) \\
&\quad + P(HTTTH) + P(THTTH) + P(TTHTH) + P(TTTHH) \\
&= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 \\
&\quad + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5 \\
&= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} \\
&= \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} = \frac{8}{32} + \frac{8}{32} + \frac{6}{32} + \frac{4}{32} \\
&= \frac{8+8+6+4}{32} = \frac{26}{32} = \frac{13}{16} = 0.8125\,.
\end{aligned}$$

It follows that the probability that more than five tosses will be required is

$$P(> 5 \text{ tosses}) = 1 - P(\leq 5 \text{ tosses}) = 1 - \frac{13}{16} = \frac{3}{16} = 0.1875\,. \quad \blacksquare$$