MATH-CCTH 1080H – Mathematics for Everyday Life
TRENT UNIVERSITY, Winter 2018 in Peterborough

Solutions to Assignment #2
Reach for the stars!?

One of Archimedes' (*c.* 287-212 B.C.) more whimsical works is *The Sand Reckoner* in which he computes an upper bound for the number of sand grains that would be required to fill the known universe, as it was then understood. This exercise required Archimedes to extend the number system then used in the Hellenistic world to handle the large numbers involved, and it may be that showing how one could so extend was the real point. Your task in this assignment will, in part, be something similar.

The *imperial minim* is the smallest official unit of volume in the imperial system of measures; one imperial minim is $0.059193880208\dot{3}\ mL$. (The minim of the United States customary system of measures is slightly larger at $0.061611519921875\ mL$.)

NOTE: Since some of the numbers that will be come up in the solutions below are very large, it will be convenient to use "scientific notation" rather than write them out in decimal form. A brief description of how this works is appended to the solution. Since a lot of the numbers are also a bit uncertain, they will be cut off at four or five significant digits at each stage of the calculation.

1. Compute the number of minims required to fill the solar system we live in. This will require you to make one of several reasonable choices as to what the boundary of the solar system ought to be. *[7]*

SOLUTION. The first thing to do, as noted in the problem, will be to decide what to use for the boundary of the solar system. The smallest reasonable choice would be the orbit of the outermost known major planet, Neptune, which has a semi-major axis of about 30.10 AU or about 4.503 billion kilometers, as noted in [1]. (1 AU, *i.e.* one astronomical unit, is the distance from the Earth to the sun, or about eight light-minutes.) The largest reasonable choice would the smallest distance from the sun at which the sun's gravitational influence is greater than that of other stars, which has been estimated to be about two light years [1]. In what follows, we will use something closer to the former, namely the distance from the sun to the outer edge of the Kuiper Belt, a ring of objects whose orbits are mostly beyond the orbit of Neptune, which distance is estimated to be about 50 AU or about 7.480 billion kilometers (*i.e.* $7,480,000,000\ km = 7.480 \times 10^9\ km$) [1].

We will also take the shape of the solar system to be a sphere. This is an easy reasonable choice – assuming it to be a cube is not that reasonable, and assuming it to be an ellipsoid that is a flattened sphere may be reasonable, but is harder to decide on parameters for. (Just how much is it reasonable to flatten the sphere?) The formula for the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Recalling that minims were defined in terms of $mL = cm^3$, we convert the distance measurement – *i.e.* the radius of our solar system sphere – to cm before using it. Since there are $1000 = 10^3\ m$ per km, $7.480 \times 10^9\ km = 7.480 \times 10^9 \times 1000\ m = 7.480 \times 10^{12}\ m$, and since there are $100 = 10^2\ cm$ per m, $7.480 \times 10^{12}\ m = 7.480 \times 100 \times 10^{12}\ cm = 7.480 \times 10^{14}\ cm$. It follows that the volume of the solar system is:

$$\begin{aligned} V &= \frac{4}{3}\pi\left(7.480 \times 10^{14}\right)^3\ cm^3 \approx \frac{4}{3} \cdot 3.1416 \cdot 418.5 \times 10^{14\cdot 3}\ cm^3 \\ &\approx 4.188 \cdot 418.5 \times 10^{42}\ cm^3 = 4.188 \cdot 4.185 \times 10^{44}\ cm^3 \\ &\approx 17.53 \times 10^{44}\ cm^3 = 1.753 \times 10^{45}\ cm^3 = 1.753 \times 10^{45}\ mL \end{aligned}$$

Since there are about $\frac{1}{0.05919} \approx 16.89$ imperial minims per ml, it follows that the volume of the solar system is approximately $16.89 \cdot 1.753 \times 10^{45}\ min \approx 29.61 \times 10^{45}\ min = 2.961 \times 10^{46}\ min$. Whew! ■

2. Compute the density of the solar system in g/L. *[3]*

SOLUTION. The vast majority of the mass of the solar system is in the sun. The estimate given in [1], using pretty conservative estimates for the total mass of the material beyond the orbit of Neptune, works out to over 99.8% of the total mass being in the sun, but even if there is a large sub-stellar object in a very distant orbit (check out the "Nemesis" hypothesis), it is a very good bet that the sun amounts to over 98% of the total mass of the solar system. Just to keep it simple, we will use the mass of the sun in place of the total mass of the solar system. This gives:

$$\begin{aligned}\text{mass of the solar system} &\approx 1 \text{ solar mass} \approx 332,900 \text{ Earth masses}\\ &\approx 332,900 \cdot 5.972 \times 10^{24}\ kg \approx 1.998 \times 10^{30}\ kg\\ &= 1000\ g/kg \cdot 1.998 \times 10^{30}\ kg = 1.998 \times 10^{33}\ g\end{aligned}$$

From the solution to question 1, we have a value of $1.753 \times 10^{45}\ mL$ for the volume of the solar system; converting this to litres gives $0.001\ L/mL \cdot 1.753 \times 10^{45}\ mL = 1.753 \times 10^{42}\ L$. Thus the density of the solar system is approximately:

$$\frac{\text{mass}}{\text{volume}} = \frac{1.998 \times 10^{33}\ g}{1.753 \times 10^{42}\ L} \approx 1.140 \times 10^{-9}\ g/L = 0.000000001140\ g/L \quad ■$$

REFERENCES

1. *Solar System*, *Wikipedia*, `en.wikipedia.org/wiki/Solar_System`, accessed 2018.01.29.
2. *Earth*, *Wikipedia*, `en.wikipedia.org/wiki/Earth`, accessed 2018.01.29.

SCIENTIFIC NOTATION. When numbers get really big or really small it becomes inconvenient to write them out fully in decimal notation, so scientists and engineers often resort to "scientific notation" writing them as a number ≥ 1 and < 10 multiplied by a suitable power of 10. For example, $7,480,000,000 = 7.480 \times 10^9$ and $0.05919 = 5.919 \times 10^{-2}$. Basically, the exponent of the 10 tells you how many places to move the decimal point to the right (if the exponent is positive) or to the left (if the exponent is negative) to convert the number to decimal form.

Besides allowing many numbers to be written more compactly and making it easier to tell the scale of a number at a glance, scientific notation has some computational advantages. Exponents add when multiplying and subtract when dividing, so, for example, $1.2 \times 10^3 \cdot 2.1 \times 10^2 = 1.2 \cdot 2,1 \times 10^{3+2} = 2.52 \times 10^5$, which certainly beats trying to multiply $1200 \cdot 210$ by hand.