Math 260 Lecture Notes

MacEwan University

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Contents

Chapter 1

Mathematical Reasoning

1.1 Statements

Definition 1.1.1 *A* **statement** *is a sentence which is either true or false, but not both simultaneously.*

Here are some examples.

Example 1.1.1 *Today is Tuesday.*

This is a statement which is true.

Example 1.1.2 *Toronto is the capital of Canada.*

This is a statement which is false.

Example 1.1.3 *I have a cat.*

This is also a statement. Note that for this statement, you know it is either true or false, even if you do not know which is the case.

Important!: "Titanic is the best movie either" is not a statement, it is an opinion.

We usually use letters like p, q, r, s to denote statements. If we write

$$p: \text{ Today is Tuesday.}$$

it means that whenever when we refer to p, we mean the statement "Today is Tuesday".

In the next few sections we will be interested in statements in general, how to combine statements, we will not address the question of whether a certain statement is true or not, until we learn truth tables.

1.1.1 Negating Statements

If we have a statement, by the **negation** of that statement we mean the complete opposite of that statement.

For example, the negation of "Today is Tuesday" is "Today is not Tuesday".

The negation of a statement p is denoted by $\sim \mathbf{p}$.

Note that the same statement can be negated in different ways, for example "It is not true that today is Tuesday" is also a negation of "Today is Tuesday". In general, when we refer to a statement, we don't mean the exact phrasing, it is the meaning which is important.

Exercise 1.1.1 *If*

$$p: \textit{ Toronto is the capital of Canada.}$$

what is $\sim p$*?*

Solution:

$$\sim p: \text{ Toronto is not the capital of Canada.}$$

Exercise 1.1.2 *If*

$$p: \textit{ I have a cat.}$$

what is $\sim p$*?*

Solution:

$$\sim p: \text{ I do not have a cat.}$$

Question 1.1.1 *What happens if we negate a question twice?*

Solution: We get the initial statement. $\sim\sim p$ is the same thing as p, we can write this answer also as

$$\sim\sim p = p\,.$$

1.1.2 Quantified Statements

A quantified statement is a statement containing the words **ALL, SOME** or **NONE**.

Example 1.1.4

$p:$ *All penguins are black*

$q:$ *Some cats are yellow*

$r:$ *Some cars are not red*

$s:$ *No students are lazy*

A quantified statement has one of four forms:

$$\text{All A are B} \tag{1.1}$$

The meaning of this statement is that every single element of A is B. Or equivalently that there is no A which is not B. "All penguins are black", means that every single penguin is black. Again, don't forget that we don't worry yet if this is a true or false statement.

$$\text{Some A are B} \tag{1.2}$$

The meaning of this statement is that there exists at least one element in A which is B. "Some cats are yellow" means that one or more cats are yellow.

Important!: In mathematics, some means anything between one and all, inclusive. Sometimes in English by "some" it is understood "some but not all", in mathematics some could mean all. Same way, by "some" it is often understood multiple, in mathematics it could also mean one.

If a lady has 8 cats and I say "some" of her cats, some can mean any of the following numbers: $1, 2, 3, 4, 5, 6, 7, 8$.

$$\text{Some A are not B} \tag{1.3}$$

The meaning of this statement is that there exists at least one element in A which is not B. "Some cars are not red" means that there is at least one car which is not red.

$$\text{No A are B} \tag{1.4}$$

The meaning of this statement is that you cannot find any element of A which is B. "No students are Lazy" means that there exists no lazy student.

1.1.3 Negating quantified statements

If

$$p: \text{ All penguins are black}$$

then

$$\sim p: \text{ Some penguins are not black}$$

Important!: Be careful when you negate quantified statements, "All penguins are not black" is NOT the negation of "All penguins are black".

When we negate a quantified statement, the quantifier changes. Not changing the quantifier is one of the most common mistakes people make when negating statements.

Question 1.1.2 *What is the negation of*

Some penguins are not black

Solution: We already seen before that if we negate a statement once, and then negate it again, we get our original statement. The negation of this statement is

$$p: \text{ All penguins are black}$$

Example 1.1.5

$$q: \textit{ Some cats are yellow}$$

What is $\sim q$*?*

Solution:

$$\sim q: \text{ No cats are yellow}$$

And exactly as before the negation of "No cats are yellow" is "Some cats are yellow".

The four quantified statements come in two groups of two. In each group, the negation of one statement is the other statement:

$$\begin{aligned} \text{All A are B} &\longleftrightarrow \text{ Some A are not B} \\ \text{Some A are B} &\longleftrightarrow \text{ No A are B} \end{aligned}$$

1.1.4 Connecting statements

Definition 1.1.2 *A* **simple statement** *is a statement which conveys a single idea and no connecting words.*

We can create more complex statements by connecting simple statement by the words **OR, AND, IF .... THEN** *and* **IF AND ONLY IF**. *We will refer to these as* **connecting words**.

Example 1.1.6

Today is Tuesday and we have a class .

Is a statement created by connecting the simple statements

$$p: \textit{ Today is Tuesday },$$

$$q: \textit{ We have a class }.$$

with the connecting word **and**.

AND statements

If p, q are two statements, the statement "p and q" is denoted by $p \wedge q$.

Example 1.1.7

$$p: \text{ Today is Tuesday },$$

$$q: \text{ We have a class }.$$

What is $p \wedge q$?

Solution:

$$p \wedge q: \text{ Today is Tuesday and we have a class}$$

OR statements

If p, q are two statements, the statement "p or q" is denoted by $p \vee q$.

Example 1.1.8

$$p: \text{ I visited London },$$

$$q: \text{ I visited Paris }.$$

What is $p \vee q$?

Solution:

$$p \wedge q: \text{ I visited London or Paris}$$

Important!: In English, an "OR" statement can have two meanings. The Statement "I visited London or Paris" can mean "I visited Paris or London, but not both" (**exclusive or**), or it could mean "I visited Paris or London, or both" (**inclusive or**).

In mathematics the or is always inclusive. $p \vee q$ always means p or q or *both*.

Exercise 1.1.3

$$p: \text{ I study },$$

$$q: \text{ I pass the class. }.$$

Write $p \vee q, p \vee \sim q, \sim p \vee q, \sim p \vee \sim q$.

Solution:

$$p \vee q: \text{ I study or I pass the class.}$$

$$p \vee \sim q: \text{ I study or I don't pass the class.}$$

$$\sim p \vee q: \text{ I don't study or I pass the class.}$$

$$\sim p \vee \sim q: \text{ I domn't study or I don't pass the class.}$$

IF... THEN statements

By an "If then Statement" we mean a statement of the form "If p then q". We denote such a statement by $\mathbf{p \to q}$.

Example 1.1.9 *The statement*

If it is raining then there are clouds outside

is an if ... then statement. It can be viewed easily this way

$$\textit{If } \underbrace{\textit{it is raining}}_{p}\ \underbrace{\textit{then}}_{\longrightarrow}\ \underbrace{\textit{there are clouds outside}}_{q}$$

Note that most if...then statements can be rephrased with the quantitative ALL. For example

If a student is registered in MATH 260 then he passed MATH 160.

can also be stated as

All students registered in MATH 260 passed MATH 160.

Important!: If p and q are two simple statements, most of the times $p \to q$ and $q \to p$ are different statements. For example, if we have

$$p: \text{ It is raining },$$

$$q: \text{ There are coulds outside }.$$

then we have

$$p \to q: \text{ If it is raining then there are clouds outside },$$

while

$$q \to p: \text{ If there are clouds outside then t is raining}.$$

The first statement is true but the second statement is false.

IF AND ONLY IF statements

Sometimes when we reverse a true if... then statement we get another true if ... then statement. Instead of writing them as two separate if ... then statements, we write it as if and only denote it by $\mathbf{p \leftrightarrow q}$

Note that $p \leftrightarrow q$ means **both** $p \to q$ and $q \to p$.

Example 1.1.10

$$\underbrace{\textit{The campus is closed}}_{p}\ \underbrace{\textit{if and only if}}_{\longleftrightarrow}\ \underbrace{\textit{it is a holiday}}_{q}$$

As I mentioned before, this statement tells us two things:

- When the campus is closed, it is a holiday.
- When it is a holiday, the campus is closed.

1.1.5 Parenthesis

When we have multiple statements or operations, we use parenthesis to emphasize which statements are grouped together.

Important!: If there is no parenthesis, $\sim$ negates only the statement immediately after it.

Exercise 1.1.4

$$p: \textit{ She is rich },$$

$$q: \textit{ She is happy }.$$

Write $\sim p \vee q, \sim p \wedge q, \sim (p \vee q), \sim (p \wedge q)$.

Solution:

$$\sim p \vee q: \text{ She is not rich or she is happy.}$$

$$\sim p \wedge q: \text{ She is not rich and she is happy.}$$

$$\sim (p \vee q): \text{ She is neither rich nor happy.}$$

$$\sim (p \wedge q): \text{ It is not true that she is rich and happpy.}$$

1.1.6 Importance of connecting words.

Important!: **If and only if** is more important than **if .. then** is more important than **and, or** is more important than **not**.

In English, if there is a comma in the phrase, it means that the statements before the comma appear in a parenthesis, and the statements after the comma appear in a parenthesis.

If there are no parenthesis **we start by identifying the most important connecting word, and put the statements before and after it in parenthesis.**

Example 1.1.11 *The most important connector in* $p \to q \wedge r$ *is* $p {\to} q \wedge r$. *Therefore it should be read as*

$$p \to (q \wedge r)$$

Example 1.1.12 *Note that this is just common sense, if you see the statement*

$$\text{If } \underbrace{\text{I study hard}}_{p} \underbrace{\text{then}}_{\longrightarrow} \underbrace{\text{I will pass the class}}_{q} \underbrace{\text{and}}_{\wedge} \underbrace{\text{I will be happy}}_{q}$$

your understanding is exactly $p \to (q \wedge r)$.

Exercise 1.1.5 *Put the parentheses in the following statements*

$$p \to q \leftrightarrow r$$

$$p \wedge \sim q \to r$$

$$p \wedge q \leftrightarrow r \vee s$$

Solution:

$$(p \to q) \leftrightarrow r$$

$$(p \wedge \sim q) \to r$$

$$(p \wedge q) \leftrightarrow (r \vee s)$$

1.1.7 Exercises

Exercise 1.1.6 *: p and q are the following two statements.*

$$p: \text{ I work hard}$$

$$q: \text{ I succed}$$

Write in words $\sim p$; $\sim q$; $p \vee q$; $p \wedge q$; $p \rightarrow q$; $\sim p \rightarrow \sim q$; $q \leftrightarrow p$.

Exercise 1.1.7 *: Write the negations of the following statements*

$$p: \text{ All cats are pets}$$

$$q: \text{ Some students are education majors}$$

$$r: \text{ Some students are not business majors}$$

$$s: \text{ No atheists go to church}$$

Exercise 1.1.8 *: Place the parentheses in the right spots for the following statements.*

$$p \rightarrow q \leftrightarrow r$$

$$p \vee q \rightarrow r$$

$$p \rightarrow r \vee q$$

$$p \rightarrow q \leftrightarrow r \vee s$$

Exercise 1.1.9 *: Rewrite the statement "No student is not passing this class" without any negation and meaning the same thing.*

Exercise 1.1.10 *: Write the negation $\sim p$ of the statement*

$$p: \text{ All banannas are yellow.}$$

Exercise 1.1.11 *: Write the negation $\sim p$ of the statement*

$$p: \text{ Some problems are easy.}$$

1.2 Truth Tables

The negation of a True statement is a False statement. We can express this in a table:

p	$\sim$ p
T	F

The negation of a False statement is a True statement. We can also express this in a table:

p	$\sim$ p
F	T

We can combine the two tables in a single table, which is called the **Truth table for Negation**.

p	$\sim$ p
T	F
F	T

1.2.1 Truth Table for OR

The truth table for OR is easy to find:

p	q	p $\vee$ q
T	T	T
T	F	T
F	T	T
F	F	F

Exercise 1.2.1 *Find the Truth table for* $p \vee \sim q$.

Solution:

p	q	$\sim$ q	p $\vee \sim$ q
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

1.2.2 Truth Table for AND

The truth table for OR is easy to find:

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Exercise 1.2.2 *Find the Truth table for* $p\wedge \sim q$.

Solution:

p	q	∼ q	p ∧ ∼ q
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Exercise 1.2.3 *Find the Truth table for* $(\sim p \vee q)\wedge \sim q$.

Solution:

p	q	∼ p	∼ p ∨ q	∼ q	(∼ p ∨ q)∧ ∼ q
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	F	F
F	F	T	T	T	T

Exercise 1.2.4 *Find the Truth table for* $p\vee \sim p$.

Solution:

p	∼ p	p ∨ ∼ p
T	F	T
F	T	T

Definition 1.2.1 *A mathematical statement which is always true is called a* **tautology**.

For example $p \vee \sim p$ is a tautology.

Exercise 1.2.5 *Find the Truth table for* $p \wedge \sim p$.

Solution:

p	$\sim$ p	p $\wedge \sim$ p
T	F	F
F	T	F

Definition 1.2.2 *A mathematical statement which is always false is called a* **self contradiction**.

For example $p \wedge \sim p$ is a self contradiction.

1.2.3 Truth Table for IF ... THEN

The truth table for if.. then is the following:

p	q	p $\to$ q
T	T	T
T	F	F
F	T	T
F	F	T

Important!: In mathematics false implies anything is TRUE. The reason for this is that in a formal proof, we only worry about the proof if the given statement is true.

For example

$$\text{If } \underbrace{x > 2}_{p} \underbrace{\text{then}}_{\to} \underbrace{x^2 > 4}_{q}.$$

This is a true statement, which tells us that whenever $x > 2$ we have $x^2 > 4$. It doesn't tell us what happens when $x \leq 2$. Note that in that case p is false, and q is sometimes true and sometimes false.

Regardless the above statement is true.
Another interesting example is

$$\text{If } \underbrace{\text{it is Tuesday}}_{p} \underbrace{\text{then}}_{\to} \underbrace{\text{I go to school}}_{q}.$$

This phrase simply means that every Tuesday I go to school. This is a true phrase.

But, depending on which day it is, this TRUE statement has the form $T \to T$ (Tuesday) or $F \to T$ (Thursday) or $F \to F$ (Sunday).

1.2.4 Truth Table for IF AND ONLY IF

The truth table for IF AND ONLY IF is easy to find:

p	q	p ∨ q
T	T	T
T	F	F
F	T	F
F	F	T

As expected, IF AND ONLY IF means that both statements have the same truth value.

Exercise 1.2.6 *Write the Truth table for* $(p \vee q) \to \sim r$.

Solution:

p	q	r	∼ r	p ∨ q	$(p \vee q) \to \sim r$
T	T	T	F	T	F
T	F	T	F	T	F
F	T	T	F	T	F
F	F	T	F	F	T
T	T	F	T	T	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	F	F

Exercise 1.2.7 *Write the Truth table for* $p \to q, q \to p, \sim p \to \sim q, \sim q \to \sim p$, $p \wedge \sim q$.

Solution:

p	q	$\sim$ p	$\sim$ q	p $\to q$	q $\to p$	$\sim p \to \sim q$	$\sim q \to \sim p$	$p \wedge \sim q$
T	T	F	F	T	T	T	T	F
T	F	F	T	F	T	T	F	T
F	T	T	F	T	F	F	T	F
F	F	T	T	T	T	T	T	F

Let us observe that $p \to q$ and $\sim q \to \sim p$ have the same truth table thus they are equivalent:

$$(p \to q) \equiv (\sim q \to \sim p)\,.$$

Important!: This is the key for what we will call in the next section a proof by contrapositive.

Also, the truth table for this is exactly the opposite of the truth table for $p \wedge \sim q$. Therefore, they are also equivalent to $\sim (p \wedge \sim q)$.

Important!: This is the key for what we will call in the next section a proof by contradiction.

We also have

$$(q \to p) \equiv (\sim p \to \sim q)\,.$$

1.2.5 Exercises

Exercise 1.2.8 *Write the Truth tables for the following:*

a) $p \to \sim q$.

b) $\sim p \leftrightarrow q$.

c) $p \wedge (\sim q \vee p)$.

d) $p \leftrightarrow q \vee r$

Exercise 1.2.9 *Write the truth table for* $(p \wedge \sim q)$

Exercise 1.2.10 *: Use truth tables to determine if the following statements are equivalent:*

a) $p \vee q$ *and* $p \to (p \vee \sim q$*.*

b) $p \vee q$ *and* $\sim (\sim p \wedge \sim q)$*.*

c) $p \to q$ *and* $\sim p \vee q$*.*

Exercise 1.2.11 *: Write the truth table for* $(p \vee \sim q)$

Exercise 1.2.12 *: Write the truth table for* $(p \wedge q) \to (\sim p)$*.*

Exercise 1.2.13 *: Write the truth table for* $(p \vee q) \to (\sim p)$*.*

Exercise 1.2.14 *: Are* $p \vee q$ *and* $(\sim p) \to q$ *equivalent?*

1.3 Proofs

1.3.1 Conditional, Converse, Inverse and Contrapositive

Definition 1.3.1 *If* p *and* q *are two statements we will call the following:*

$p \to q$ *is called the* **conditional** *statement.*

$q \to p$ *is called the* **converse** *statement.*

$\sim p \to \sim q$ *is called the* **inverse** *statement.*

$\sim q \to \sim p$ *is called the* **contrapositive** *statement.*

Exercise 1.3.1 *Write the conditional, converse, inverse and contrapositive for*

p *: you are 17*

q *: you cannot vote*

Solution:

conditional : If you are 17 then you cannot vote.

converse : If you cannot vote then you are 17.

inverse : If you are not 17 then you can vote.

contrapositive : If you can vote then you are not 17.

1.3.2 Direct Proofs

A direct proof is a proof of the form $p \to q$. p is the statement given by the problem, while q is what we need to prove.

Our goal is to start at p and by reasoning to arrive at q.

Exercise 1.3.2 *If $x \geq 2$ and $y \geq 3$ show that $2x + 3y \geq 13$.*

Solution:

Given	Wanted
$x \geq 2$	$2x + 3y \geq 13$
$y \geq 3$	

For us p is the red statement, q is the blue statement, and we need to go from p to q. Since our goal is to show that $2x + 3y \geq 13$, we need to create $2x + 3y$ from the red statements. To do this, we first create $2x$ and $3y$.

$$x \geq 2 \Rightarrow 2x \geq 4$$
$$y \geq 3 \Rightarrow 3y \geq 9$$

Adding these two statements, we get $2x + 3y \geq 13$, which is exactly q. This completes the proof.

Exercise 1.3.3 *If x is even show that $x^2 + 1$ is odd.*

Solution:

Given	Wanted
x is even	$x^2 + 1$ is odd

x is even means that $x = 2k$ for some integer x. Then

$$x^2 + 1 = (2k)^2 + 1 = 4k^2 + 1 = 2(2k^2) + 1\,.$$

is an odd number.

1.3.3 Proof by Contrapositive

Recall that $(p \to q) \equiv (\sim q \to \sim p)$. Therefore

To prove a direct statement $p \to q$ we can prove instead, **if we prefer** the contrapositive statement $\sim q \to \sim p$.

Such a proof is called a **proof by contrapositive**.

To do a proof by contrapositive, we start by writing the contrapositive statement, and then we prove it. Note that for the contrapositive we do a direct proof.

Important: Whenever when we are not sure if we should do a direct proof or a proof by contrapositive, write both the direct implication $p \to q$ and the contrapositive $\sim q \to \sim p$. Whichever looks easier, that is the one we prove. **Most of the times it is easier to start at the simpler statement and go towards the more complex one, then the other way around.**

Exercise 1.3.4 *Write the contrapositive of*

If you take Math 260 then you passed Math 160.

Solution:

If you didnot pass Math 160 then you don't take Math 260.

Exercise 1.3.5 *If x^2 is odd show that x is odd.*

Solution:

The contrapositive of this statement is :

contrapositive : If x is even then x^2 is even.

As the contrapositive goes from the simpler statement to the more complex one, we will prove the contrapositive.

Given	Wanted
x is even	x^2 is even.

x is even means that $x = 2k$ for some integer x. Then

$$x^2 = (2k)^2 = 4k^2 = 2(2k^2) .$$

is an even number.

We proved the contrapositive statement, therefore we proved our statement by contrapositive.

Exercise 1.3.6 *If* $x^2 + 5x < 0$ *then* $x < 0$.

Solution:

The contrapositive of this statement is :

$$\text{contrapositive : If } x \geq 0 \text{ then } x^2 + 5x \geq 0.$$

As the contrapositive goes from the simpler statement to the more complex one, we will prove the contrapositive.

Given	Wanted
$x \geq 0$	$x^2 + 5x \geq 0$

We know that $x^2 \geq 0$. As $x \geq 0$ we also have $5x \geq 0$. Adding them together, we get $x^2 + 5x \geq 0$.

1.3.4 Proofs by contradiction

Recall that $p \to q$ and $p \wedge \sim q$ have opposite truth tables. Therefore, to show that $p \to q$, we can prove instead that $p \wedge \sim q$ is a selfcontradiction.

A proof by contradiction starts by assuming that $\sim q$. Then, combining p and $\sim q$, our aim is to get a contradiction. We complete by proof, by observing that since we got a contradiction, our assumption is wrong, therefore q must be true.

Important: Contradiction is often useful when the problem asks us to show that something doesn't happen. Whenever when we see such a problem, we should consider a proof by contradiction. A proof by contradiction doesn't always work in that situation, but works more often than not.

Exercise 1.3.7 *Show that we cannot find positive integers* x, y *so that*

$$x^2 = 2 + y^2 .$$

Solution: Since we have a negative statement, we try a proof by contradiction.

We assume by contradiction that we can find positive integers x, y **so that**

$$x^2 = 2 + y^2 .$$

Given	Wanted
x, y integers	contradiction
$x^2 = 2 + y^2$	

We know

$$2 = x^2 - y^2 = (x - y)(x + y)\,.$$

The only way of writing 2 as a product of two positive integers is $1 \cdot 2$. As $x - y < x + y$ we have

$$x - y = 1$$
$$x + y = 2$$

Adding the two relations we get $2x = 2$ or $x = \frac{3}{2}$.

Therefore we know that x is a positive integer, and that $x = \frac{3}{2}$. This is a **contradiction.**

Since we got a contradiction, our assumption is wrong. Therefore, we showed that we cannot find positive integers x, y so that

$$x^2 = 2 + y^2\,.$$

Note: Any problem which we can solve by contrapositive, also works by contradiction. In this case, a proof by contrapositive is shorter. The converse is not true, many proofs by contradiction don't work by contrapositive.

Exercise 1.3.8 *Show that $\sqrt{2}$ is irrational.*

Solution: Note that the only thing we know about irrational numbers is that they are not rational. Therefore the problem asks us to show that $\sqrt{2}$ is not rational.

Since we have a negative statement, we try a proof by contradiction.

We assume by contradiction $\sqrt{2}$ is rational.

Given	Wanted
$\sqrt{2}$ is rational	contradiction

Since $\sqrt{2}$ is rational, we can write it as a **reduced** fraction

$$\sqrt{2} = \frac{m}{n}$$

with m, n integers. Then

$$2 = \frac{m^2}{n^2}$$

and

$$2n^2 = m^2 .$$

The left hand side is even, therefore the right hand side is also even. Since m^2 is even, m must be even.

Let $m = 2k$. Then

$$2n^2 = (2k)^2 = 4k^2 \Rightarrow n^2 = 2k^2 .$$

We repeat the argument: The right hand side is even, therefore the left hand side is also even. Since n^2 is even, n must be even.

Therefore, we know that m, n are both even, which means that $\frac{m}{n}$ can be simplified by an 2. We also know that $\frac{m}{n}$ is reduced, which means it cannot be reduced. These two statements contradict eachother, therefore we reached a **contradiction.**

Since we got a contradiction, our assumption is wrong. Therefore, we showed $\sqrt{2}$ is not rational.

1.3.5 Exercises

Exercise 1.3.9 *If*

$$p : \textit{ I live in Edmonton}$$

and

$$q : \textit{ I live in Alberta}$$

write the conditional $p \to q$*, the converse* $q \to p$*, the inverse* $\sim p \to \sim q$ *and the contrapositive* $\sim q \to \sim p$*.*

Exercise 1.3.10 *: Write the converse, inverse and contrapositive for the following statement*

"If you work hard you will succeed."

Exercise 1.3.11 : *What is the converse of the inverse of* $p \to q$*?*

Exercise 1.3.12 : *If* $x \geq 1$ *and* $y \geq 2$ *show that* $2x + 3y \geq 8$.

Exercise 1.3.13 : *If* $x^2 < 1$ *prove by contrapositive that* $x < 1$.

Exercise 1.3.14 : *Prove by contradiction that* $\sqrt{3}$ *is not rational.*

Exercise 1.3.15 : *Write the converse, inverse and contrapositive for*

$$q: \text{ If it rains then I need my umbrella.}$$

Exercise 1.3.16 : *Write the converse, inverse and contrapositive for*

$$q: \text{ If we have an exam then it is Thursday.}$$

Exercise 1.3.17 :*Prove by contrapositive that if* $x^2 + 2x < 0$ *then* $x < 0$.

Exercise 1.3.18 : *Prove by contrapositive that if* $x^2 + 1$ *is even then* x *is odd.*

Exercise 1.3.19 : *Prove that if* $n^2 + 1$ *is odd then* n *is even.*

Exercise 1.3.20 : *If* $x \geq 2$ *and* $y \geq 3$ *prove that* $2x + y \geq 7$.

Chapter 2

Real numbers

2.1 Real Numbers

In Math 160 we first started by studying the Natural numbers. The set of natural numbers is usually denoted by $\mathbb{N}$. The natural numbers are:

$$1, 2, 3, 4, ...$$

Later we discovered the integers. The set of integers is usually denoted by $\mathbb{Z}$. The integers are:

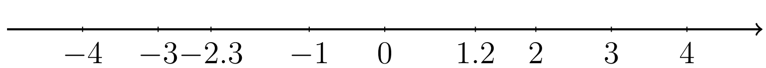
$$... - 4, -3, -2, -1, 01, 2, 3, 4, ...$$

Division leads to fractions, and any ratio of two integers, the denominator being not zero, is called a rational number. The set of all rational numbers is denoted by $\mathbb{Q}$. Numbers like $\frac{1}{2}, \frac{3}{5}, \frac{-1}{2}, 3$ are rational numbers.

But we have seen in the previous section that some numbers like $\sqrt{2}$ cannot be rational. These are called irrational numbers. Other examples of irrational numbers are $\sqrt{3}, \sqrt{2} + \sqrt{3}, \pi$.

Rational and irrational numbers form together the real numbers. real numbers are denoted by $\mathbb{R}$.

Real numbers can be represented as points on the real line:



2.1.1 Intervals

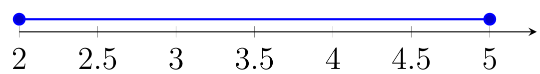
An interval consists of all numbers between two fixed real numbers.

We use round brackets $(,)$ if the end point(s) is not in the interval and square brackets $[,]$ if the end point(s) is in the interval.

For example $[2,5]$ represents all numbers between 2 and 5 inclusive. This interval can be represented by the inequality

$$2 \leq x \leq 5\,.$$

It can also be drawn on the real axis:

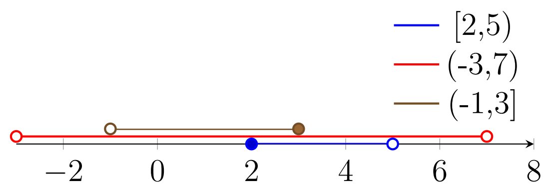


Important: When we draw an interval on the axis, we use hollow dot to emphasize that the end point is not in the interval, and full dot to emphasize that the end point is in the interval.

Exercise 2.1.1 *Express the intervals* $[2,5)$ *;* $(-3,7)$ *and* $(-1,3]$ *as inequalities and draw them on the real axis.*

Solution:

$$\begin{aligned} [2,5): &\qquad 2 \le x < 5 \\ (-3,7): &\qquad -3 \le x < 7 \\ (-1,3]: &\qquad -1 < x \le 3 \end{aligned}$$



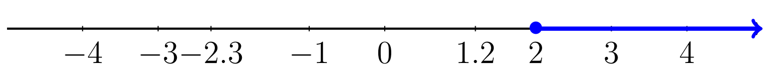
To denote all the numbers greater than a real number a we use the notation (a,∞) or $[a,\infty)$ depending if a is included or not. Same way, to denote all the numbers less than a real number a we use the notation $(-\infty,a)$ or $(-\infty,a]$ depending if a is included or not.

Important: Since ∞ and $-\infty$ are not numbers, they are never included in the interval. **The interval is always open at ∞ and $-\infty$**.

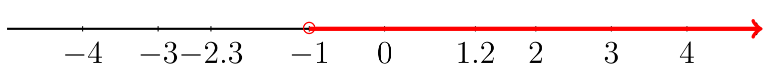
Exercise 2.1.2 *Express the intervals* $[2,\infty)$ *;* $(-1,\infty)$ *and* $(-\infty,3]$ *as inequalities and draw them on the real axis.*

Solution:

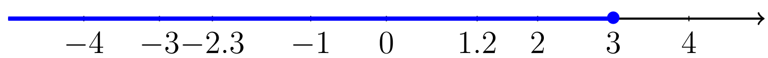
$$[2,\infty): \qquad 2 \le x\,.$$



$$(-1,\infty): \qquad -1 < x\,.$$



$$(-\infty, 3]: \qquad x \leq 3 \, .$$



2.1.2 Intersection

In Mathematics we use the notation $\{,\}$ to refer to the elements listed in that list. Such a list is called a **set**.

Example 2.1.1

$$\{1, 2, 5\} \, .$$

means the set consisting of the numbers 1, 2 and 5.

Important: $\{1, 2\}$ means only the numbers 1 and 2, while $[1, 2]$ means all the numbers between 1 and 2 inclusive.

We also use the symbol $\emptyset$ to refer to an empty list. This symbol is called the **empty set**, and is always used to say that there is no number like the one we seek.

Definition 2.1.1 *The* **intersection** *of two sets consists of all common elements. The intersection is denoted by* $\cap$.

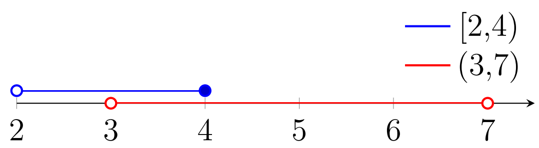
Example 2.1.2

$$\{1, 2, 3, 5\} \cap \{1, 2, 4, 6\} = \{1, 2\} \, .$$

Note: The intersection consists of all elements in the first set **and** in the second set. This is why the symbols for intersection ($\cap$) and "and" ($\wedge$) are similar.

Exercise 2.1.3 *Find* $(2, 4] \cap (3, 7)$.

Solution:

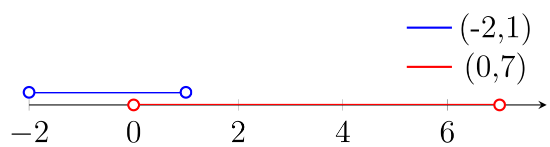


In the intersection we seek the points colored with both colors. Thus

$$(2,4] \cap (3,7) = (3,4]\,.$$

Exercise 2.1.4 *Find* $(-2,1) \cap (0,7)$.

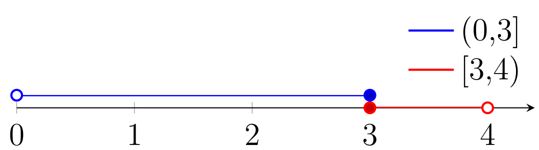
Solution:



$$(-2,1) \cap (0,7) = (0,1)\,.$$

Exercise 2.1.5 *Find* $(0,3] \cap [3,4)$.

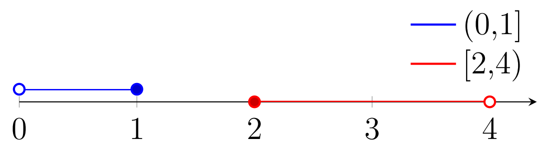
Solution:



$$(0,3] \cap [3,4) = \{3\}\,.$$

Exercise 2.1.6 *Find* $(0,1] \cap [2,4)$.

Solution:

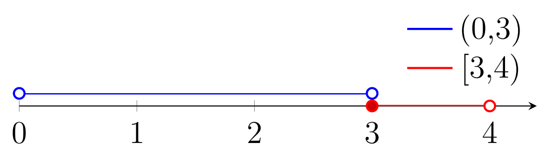


There is no point colored with both colors, we write $\emptyset$ to say this.

$$(0,1] \cap [2,4) = \emptyset \,.$$

Exercise 2.1.7 *Find* $(0,3) \cap [3,4)$.

Solution:



$$(0,3) \cap [3,4) = \emptyset \,.$$

Important: When we intersect intervals, there are three possible answers: a smaller interval, a point or the empty set.

2.1.3 Union

Definition 2.1.2 *The* **union** *of two sets consists of all elements of the two sets. The intersection is denoted by* $\cup$.

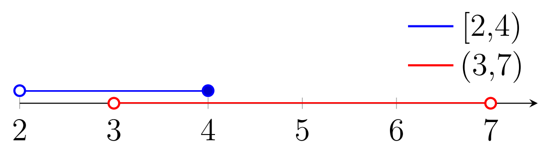
Example 2.1.3

$$\{1,2,3,5\} \cup \{1,2,4,6\} = \{1,2,3,4,5,6\} \,.$$

Note: The union consists of all elements in the first set **or** in the second set. This is why the symbols for intersection ($\cup$) and "and" ($\vee$) are similar.

Exercise 2.1.8 *Find* $(2,4] \cup (3,7)$.

Solution:

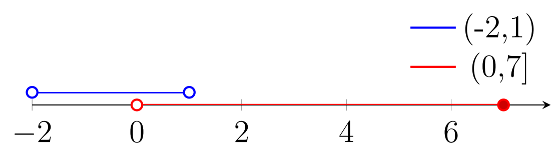


In the union we seek the points colored with one or two colors. Thus

$$(2, 4] \cup (3, 7) = (2, 7)\,.$$

Exercise 2.1.9 *Find* $(-2, 1) \cup (0, 7]$.

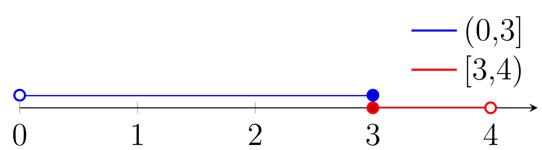
Solution:



$$(-2, 1) \cup (0, 7) = (-2, 7]\,.$$

Exercise 2.1.10 *Find* $(0, 3] \cup [3, 4)$.

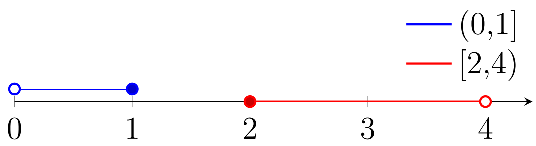
Solution:



$$(0, 3] \cup [3, 4) = (0, 4]\,.$$

Exercise 2.1.11 *Find* $(0, 1] \cup [2, 4)$.

Solution:

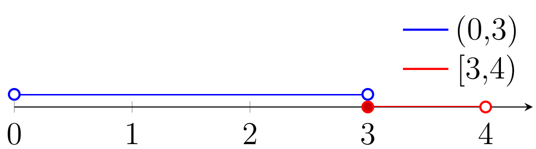


There is no way of writing the answer in a better form.

$$(0,1] \cup [2,4) = \emptyset \,.$$

Exercise 2.1.12 *Find* $(0,3) \cup [3,4)$.

Solution:



$$(0,3) \cup [3,4) = (0,4) \,.$$

Important: When we take the union of intervals, if the two intervals have one or more common points we get a larger interval. If they don't have any common points, we cannot express our answer better than an union of two intervals.

2.1.4 Absolute Value

Any real number can be thought as consisting of two parts: a non-negative number (i.e positive or zero) and a sign + or − in front. The non-negative part of it is called "the absolute value". Mathematically it is defined as follows:

Definition 2.1.3 *The* **absolute value** *of o real number a is the value $|a|$ defined as*

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases} \,.$$

Note that if a is a positive number, then $|a|$ is exactly a. If a is negative, it already has a minus inside. The absolute value $|a|$ is then the number without the minus. The definition simply erases this minus by adding a second one: two minuses cancel eachother.

Example 2.1.4 *Find*

i) $|2-3|$

ii) $|3-\sqrt{5}|$

iii) $|5-|3-9||$.

Solution
i)

$$|2-3| = |-1| = 1\,.$$

ii) as $\sqrt{5} > \sqrt{4} = 2$ it follows that $2-\sqrt{5}$ is negative.

$$|2-\sqrt{5}| = -(2-\sqrt{5}) = -2+\sqrt{5} = \sqrt{5}-2\,.$$

iii)

$$|5-|3-9|| = |5-|-6|| = |5-6| = |-1|1\,.$$

Example 2.1.5 *Calculate*

i) $-|-3|$

ii) $\frac{|2-5|}{2-5}$

Solution
i)

$$-|-3| = -3\,.$$

ii)

$$\frac{|2-5|}{2-5} = \frac{3}{-3} = -1\,.$$

Note that the absolute value can be used to find the distance between points on the real axis.

Theorem 2.1.1 *The distance between a and b is $|b-a|$.*

Example 2.1.6 *Find the distance between 2 and -5.*

The distance is $|2-(-5)| = |2+5| = 7$.

Example 2.1.7 *Find the distance between 2 and $\sqrt{5}$.*

The distance is $|2-\sqrt{5}| = -(2-\sqrt{5}) = \sqrt{5}-2$.

Property 2.1.1 *The absolute value has the following properties:*

i) $|ab| = |a| \cdot |b|$.

ii) $|\frac{a}{b}| = \frac{|a|}{|b|}$.

iii) $|-a| = |a|$.

2.1.5 Exercises

Exercise 2.1.13 *Draw the following intervals on the real axis:*

- $[-2, 1)$
- $(1, 7)$
- $(-2, \infty)$.

Exercise 2.1.14 *Find the following:*

- $[-2, 1) \cup (0, 3)$
- $(1, 7) \cap [0, 2)$
- $(-\infty, 2] \cap [2, \infty)$.
- $(1, 3) \cap (3, 6)$.

Exercise 2.1.15 *Calculate*

i) $|2-\sqrt{5}| =$

ii) $\frac{|3-\sqrt{7}|}{4-|2-3|} =$

iii) $|4 - |5 - 11|| =$

Exercise 2.1.16 *a) Find* $(2, 7) \cap (3, 9]$.

b) Find $(1, 4] \cap [4, 7)$.

Exercise 2.1.17 *For which value of a we have* $(1, 3) \cap (a, 7] = (2, 3)$*?*

Exercise 2.1.18 *Calculate*

$$|\sqrt{5} - 3| + 2\,.$$

Exercise 2.1.19 *a) Find* $(1, 4) \cup (3, 9]$.

b) Find $(1, 5] \cap [3, 5)$.

Exercise 2.1.20 *Calculate*

$$|\sqrt{7} - 4| - 1\,.$$

Exercise 2.1.21 *Find all numbers a so that*

$$|a - 1| + |2 - a| = 1\,.$$

2.2 Powers and Radicals

2.2.1 Powers

If a is a real number and n is a positive integer, we denote by a^n the number

$$a^n = \underbrace{a \cdot a \cdot a \cdot ... \cdot a}_{n \text{ times}}\,.$$

Example 2.2.1

$$2^3 = 2 \cdot 2 \cdot 2 = 8\,.$$

Example 2.2.2

$$(-3)^3 = (-3) \cdot (-3) \cdot (-3) = -27\,.$$

If $a \neq 0$ we define

$$a^0 = 1\,,$$

and

$$a^{-n} = \frac{1}{a^n}\,,$$

Important: 0^0 doesn't make any sense. Same way, 0 to a negative power means division by zero, which again makes no sense.

Example 2.2.3

$$5^0 = 1\,.$$

Example 2.2.4

$$(-2)^{-2} = \frac{1}{(-2)^2} = \frac{1}{4}\,.$$

Property 2.2.1 *If a is a real number and m, n are integers then*

$$a^{m+n} = a^m \cdot a^n\,.$$

Proof :

If m, n are both positive, then

$$a^m a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ times}} \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ times}} = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m+n \text{ times}} = a^{m+n}\,.$$

Now, if m or n is zero or negative, the claim follows from the choice we made for a^0 and a^{-n}.

□

Note: Because Property 2.2.1, the choice $a^0 = 1$ is the only one which makes sense. Indeed

$$a^n \cdot a^0 = a^{n+0} = a^n\,.$$

implies that a^0 should be 1.

Same way

$$a^n a^{-n} = a^{n-n} = a^0 = 1\,,$$

implies that a^{-n} should be $\frac{1}{a^n}$.

Property 2.2.2 *If a is a real number and m, n are integers then*

$$a^{m-n} = \frac{a^m}{a^n}\,.$$

Proof:

If m, n are both positive and $m > n$, then

$$\frac{a^m}{a^n} = \frac{a \cdot a \cdot a \cdot \ldots \cdot a}{a \cdot a \cdot a \cdot \ldots \cdot a} = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m-n \text{ times}} = a^{m-n}\,.$$

Exactly as before, the other cases follow from the choices we made for a^0 and a^{-n}.

□

A faster proof is by observing that by Property 2.2.1 we have

$$a^{m-n} \cdot a^n = a^m\,.$$

Dividing both sides by a^n we obtain this property.

Property 2.2.3 *If a is a real number and m, n are integers then*

$$(a^m)^n = a^{mn}\,.$$

Proof:

If m, n are both positive, then

$$(a^m)^n = \underbrace{\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ times}}\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ times}} \cdot \ldots \cdot \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ times}}}_{n \text{ times}} = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{mn \text{ times}} = a^{mn}\,.$$

Again, the other cases follow from the choice we made.

□

Property 2.2.4 *If a is a real number and n is integer then*

$$(ab)^n = a^n b^n\,.$$

Proof:

If n is positive, then

$$(ab)^n = \underbrace{(ab) \cdot (ab) \cdot (ab) \cdot ... \cdot (ab)}_{n \text{ times}} = \underbrace{a \cdot a \cdot a \cdot ... \cdot a}_{n \text{ times}} \cdot \underbrace{b \cdot b \cdot b \cdot ... \cdot b}_{n \text{ times}} = a^n b^n \,.$$

□

Property 2.2.5 *If a is a real number and n is integer then*

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \,.$$

Proof:

If n is positive, then

$$(\frac{a}{b})^n = \underbrace{(\frac{a}{b}) \cdot (\frac{a}{b}) \cdot (\frac{a}{b}) \cdot ... \cdot (\frac{a}{b})}_{n \text{ times}} = \frac{a \cdot a \cdot a \cdot ... \cdot a}{b \cdot b \cdot b \cdot ... \cdot b} = \frac{a^n}{b^n} \,.$$

□

Example 2.2.5

$$2^4 \cdot (2^2)^3 = 2^4 \cdot 2^6 = 2^{4+6} = 2^{10} \,.$$

Example 2.2.6

$$\frac{2^2 \cdot (2^{-2})^3}{2^2 \cdot (2^3)^{-1}} = \frac{2^2 \cdot 2^{-6}}{2^2 \cdot 2^{-3}} = \frac{2^{-4}}{2^{-1}} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8} \,.$$

Example 2.2.7

$$\frac{6^3(2^{-1}3^{-2})^{-3}}{2^3 3^2} = \frac{2^3 3^3 2^3 3^6}{2^3 3^2} = \frac{2^6 3^9}{2^3 3^2} = 2^3 3^7 \,.$$

Example 2.2.8

$$\frac{x^2 x^{-3}}{x^{-4}} = \frac{x^{-1}}{x^{-4}} = x^3 \,.$$

Example 2.2.9

$$(2x^2)^2 = 2^2 (x^2)^2 = 4x^4 \,.$$

Example 2.2.10

$$\left(\frac{x^2 y (x^{-1} y^{-2})^{-3}}{x^2 y^2 (xy)^{-1}}\right)^2 = \left(\frac{x^2 y x^3 y^6}{x^2 y^2 x^{-1} y^{-1}}\right)^2 = \left(\frac{x^5 y^7}{xy}\right)^2 = \left(x^4 y^6\right)^2 = x^8 y^6 \,.$$

2.2.2 Radicals

If $n \geq 2$ is even and $a \geq 0$ the number $\sqrt[n]{a}$ denotes the positive number whose n^{th} power is a.

Intuitively

$$\sqrt[n]{a} = b \quad \text{means} \quad b^n = a\,.$$

Because it is the most frequent radical, $\sqrt[2]{\ }$ is denoted simply $\sqrt{\ }$.

Important: Whenever when question asks what is $\sqrt[n]{a}$, remember that the question is really asking: which number to the n^{th} power is equal to a?

Example 2.2.11 *Find* $\sqrt{36}$.

Solution:

Since $6^2 = 36$ we have

$$\sqrt{36} = 6\,.$$

Example 2.2.12 *Find* $\sqrt[4]{16}$.

Solution: Since $16 = 2^4$ we have

$$\sqrt[4]{16} = 2\,.$$

Important: One of the most common mistakes in mathematics is $\sqrt{x^2} = x$. While this is true for positive numbers, it is **NOT true** for negative values of x. The right formula is

$$\sqrt{x^2} = |x|\,.$$

Same formula holds for all **even** roots.

$$\sqrt[2n]{x^{2n}} = |x|\,.$$

Example 2.2.13 *Find* $\sqrt{(-4)^2}$.

Solution: Since $(-4)^2 = 16$ we have

$$\sqrt{(-4)^2} = \sqrt{16} = 4\,.$$

If $n \geq 2$ is odd and a is any real number then the number $\sqrt[n]{a}$ denotes the number whose n^{th} power is a.

Important: Even radicals only make sense if the number under the radical is not negative. Even radicals cannot be negative.

Odd radicals make sense for all numbers, and they can be positive and negative, but not both at once.

Example 2.2.14 *Find* $\sqrt[3]{27}$.

Solution: Since $3^3 = 27$ we have

$$\sqrt[3]{27} = 3\,.$$

Example 2.2.15 *Find* $\sqrt[5]{32}$.

Solution: Since $2^5 = 32$ we have

$$\sqrt[5]{32} = 2\,.$$

Example 2.2.16 *Find* $\sqrt[3]{-8}$.

Solution: Since $(-2)^3 = -8$ we have

$$\sqrt[3]{-8} = -2\,.$$

Example 2.2.17 *Find* $\sqrt[4]{-16}$.

Solution: Since 4 is even $\sqrt[4]{-16}$ **doesn't make sense**.

We are now ready to define rational powers. Lets figure out what $a^{\frac{m}{n}}$ should be:

$$\begin{aligned} a^{\frac{m}{n}} &= ?? \\ \left(a^{\frac{m}{n}}\right)^n &= (??)^n \\ a^{\frac{m}{n}\cdot n} &= (??)^n \\ a^m &= (??)^n \end{aligned}$$

This tells us that ?? to the n^{th} power should be a^m, which is exactly the definition of $\sqrt[n]{a^m}$.

Definition 2.2.1 *Let a be a real number and m, n integers with $n > 0$. We define*

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}\,.$$

Important:

$$\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m\,.$$

Example 2.2.18 *Find $27^{\frac{2}{3}}$.*

Solution:

$$27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9\,.$$

Property 2.2.6 *The roots satisfy the following three properties:*

$$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}\,.$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}\,.$$

$$\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}\,.$$

Important: In general

$$\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}\,.$$

Note that we can write

$$\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}\,.$$

We can always pull perfect squares from under the square root. This idea is important in exercises as the following:

Example 2.2.19 *Find $2\sqrt{12} - 3\sqrt{27} + 3\sqrt{48}$.*

Solution:

$$2\sqrt{12} - 3\sqrt{27} + 3\sqrt{48} = 2\sqrt{4 \cdot 3} - 3\sqrt{9 \cdot 3} + 3\sqrt{16 \cdot 3}$$
$$= 2 \cdot 2\sqrt{3} - 3 \cdot 3\sqrt{3} + 3 \cdot 4\sqrt{3} = 4\sqrt{3} - 9\sqrt{3} + 12\sqrt{3} = 7\sqrt{3}$$

Example 2.2.20 *Find* $2\sqrt{2} - 5\sqrt{8} + 4\sqrt{18}$.

Solution:

$$
\begin{aligned}
2\sqrt{2} - 5\sqrt{8} + 4\sqrt{18} &= 2\sqrt{2} - 5\sqrt{4 \cdot 2} + 4\sqrt{9 \cdot 2} \\
&= 2\sqrt{2} - 5 \cdot 2\sqrt{2} + 4 \cdot 3\sqrt{3} = 2\sqrt{2} - 10\sqrt{2} + 12\sqrt{2} = 4\sqrt{2}
\end{aligned}
$$

Example 2.2.21 *Find* $2\sqrt{3} + \sqrt{27} - \sqrt{75}$.

Solution:

$$
\begin{aligned}
2\sqrt{3} + \sqrt{27} - \sqrt{75} &= 2\sqrt{3} + \sqrt{9 \cdot 3} - \sqrt{25 \cdot 3} \\
&= 2\sqrt{3} + 3\sqrt{3} - 5\sqrt{3} = 0
\end{aligned}
$$

Example 2.2.22 *Find* $\sqrt{2}\sqrt[3]{2}\sqrt[4]{2}$.

Solution:

$$
\begin{aligned}
\sqrt{2}\sqrt[3]{2}\sqrt[6]{2} &= 2^{\frac{1}{2}} 2^{\frac{1}{3}} 2^{\frac{1}{6}} \\
&= 2^{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}} = 2^{\frac{3+2+1}{6}} = 2
\end{aligned}
$$

Example 2.2.23 *Find* $\frac{\sqrt{12}}{\sqrt{27}}$.

Solution:

$$
\begin{aligned}
\frac{\sqrt{12}}{\sqrt{27}} &= \sqrt{\frac{12}{27}} = \sqrt{\frac{4}{9}} \\
&= \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}
\end{aligned}
$$

Example 2.2.24 *Find* $\left(\frac{9}{25}\right)^{\frac{1}{2}}$.

Solution:

$$
\left(\frac{9}{25}\right)^{\frac{1}{2}} = \sqrt{\frac{9}{25}} = \frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5}
$$

Example 2.2.25 *Find* $\left(\frac{4}{9}\right)^{-\frac{3}{2}}$.

Solution:

$$\left(\frac{4}{9}\right)^{-\frac{3}{2}} = \frac{1}{\left(\frac{4}{9}\right)^{\frac{3}{2}}} = \frac{1}{\left(\sqrt{\frac{4}{9}}\right)^3}$$
$$= \frac{1}{\left(\frac{\sqrt{4}}{\sqrt{9}}\right)^3} = \frac{1}{\left(\frac{2}{3}\right)^3} = \frac{1}{\frac{8}{27}} = \frac{27}{8}$$

2.2.3 Exercises

Exercise 2.2.1 *Calculate*

i) $2^2 \cdot (2^3)^4 =$

ii) $\frac{2^3(2^{-1})^2}{2^2} =$

iii) $\frac{2^3 \cdot 3^5 \cdot (2^{-2}3^2)^{-1}}{(2 \cdot 3^2)^2} =$

Exercise 2.2.2 *Calculate*

i) $(2x^2)^3 =$

ii) $\frac{x^2x^3}{(x^2)^2} =$

iii) $\frac{(x^2)^{-1}(x^{-3})^{-2}}{x(x^3)^{-2}} =$

Exercise 2.2.3 *Calculate*

i) $\sqrt[4]{16}$.

ii) $\frac{\sqrt{27}}{\sqrt{12}}$

iii) $2\sqrt{75} - 4\sqrt{12} + 3\sqrt{27}$.

Exercise 2.2.4 *Find*

$$\frac{\sqrt[3]{\sqrt[4]{2}}}{\sqrt{\sqrt[6]{2}}}.$$

Exercise 2.2.5 *Calculate*
$$\frac{2^2 \cdot 2^5}{(2^2)^2}\,.$$

Exercise 2.2.6 *Calculate*

a) $\sqrt[3]{-27} + \sqrt{9}$.

b) $2\sqrt{18} + 3\sqrt{27} - 5\sqrt{3}$.

Exercise 2.2.7 *For which value of n is*
$$\sqrt[n]{9} = \sqrt[6]{3}\,?$$

Exercise 2.2.8 *Calculate*

a) $\frac{\sqrt{18}}{\sqrt{50}}$.

b) $\sqrt{2} + 5\sqrt{8} - 3\sqrt{18}$.

Exercise 2.2.9
$$\frac{(2^{-1})^2 \cdot 2^5}{2^2}\,.$$

2.3 Linear Equations and Inequalities

2.3.1 Linear Equations

Definition 2.3.1 *An* **equation** *is a formula of the form $A = B$, where A, B are expressions containing one or more unknowns.*

By a **solution** *we mean any number or numbers which make the equation true.*

Example 2.3.1 *Are $1, 2, 3$ solutions to $x^2 - 4x + 3 = 0$?*

Solution:
$$1^2 - 4 * 1 + 3 = 0\checkmark\,,$$
1 is a solution.
$$2^2 - 4 * 2 + 3 = 0X\,,$$

2 is not a solution.

$$3^2 - 4 * 3 + 3 = 0 \checkmark ,$$

3 is a solution.

Important!: By solving an equation we understand finding **all** solutions.

Important!: Most of the times it is much easier to check the solutions than finding them. While this doesn't help us solve equations, it tells us that most of the times it takes very little to check if we got the right answer.

Definition 2.3.2 *An* **linear equation** *is an equation of the form*

$$ax + b = 0 .$$

In this section we solve linear equations, as well as equations which after few steps become linear.

Theorem 2.3.1 *If $a \neq 0$ the linear equation*

$$ax + b = 0 ,$$

has unique solution

$$x = -\frac{b}{a} .$$

Important!: Sometimes when we start with an equation, after few steps we get something like $0 = 0$, or some other true statement. This means that all x work, the solution is $x \in \mathbb{R}$.

Important!: Sometimes when we start with an equation, after few steps we get something like $1 = 0$, or some other false statement. This means that the equation has no solution.

To solve any linear equation we do the following:

Step 1: Open any bracket containing the variable x and another term.

Step 2: Move all terms containing x on one side, all the other terms on the other side.

Step 3: On its side, make x a common factor.

Step 4: Divide by the number in front of x.

Example 2.3.2 *Solve* $2x + 3 = 5$.

Solution:

$$\begin{aligned} 2x + 3 &= 5 \\ 2x &= 5 - 3 \\ 2x &= 2 \\ x &= 1 \end{aligned}$$

Example 2.3.3 *Solve* $3x + 2 = 4x + 1$.

Solution:

$$\begin{aligned} 3x + 2 &= 4x + 1 \\ 3x - 4x &= 1 - 2 \\ -x &= -1 \\ x &= 1 \end{aligned}$$

Example 2.3.4 *Solve* $3x + 2 = 4x + 1$.

Solution:

$$\begin{aligned} 2(x + 2) &= 5(x + 3) \\ 2x + 4 &= 5x + 15 \\ 2x - 5x &= 15 - 4 \\ -3x &= 11 \\ x &= \frac{11}{-3} = -\frac{11}{3} \end{aligned}$$

Example 2.3.5 *Solve* $\frac{4x-1}{3} + \frac{x}{4} = -2$.

Solution:

$$
\begin{aligned}
\frac{4x-1}{3}+\frac{x}{4} &= -2\\
\frac{16x-4}{12}+\frac{3x}{12} &= -2\\
\frac{19x-4}{12} &= \frac{-24}{12}\\
19x-4 &= -24\\
19x = -20x \quad &= -\frac{20}{19}
\end{aligned}
$$

Example 2.3.6 *Solve* $3(x+2) = 2(x+1) + x$.

Solution:

$$
\begin{aligned}
3(x+2) &= 2(x+1)+x\\
3x+6 &= 2x+2+x\\
3x+6 &= 3x+2\\
3x-3x &= 2-6\\
0 &= -4
\end{aligned}
$$

There is no solution.

Example 2.3.7 *Solve* $x+1 = \frac{2x+1}{2}$.

Solution:

$$
\begin{aligned}
x+1 &= \frac{2x+1}{2}\\
\frac{2(x+1)}{2} &= \frac{2x+2}{2}\\
2(x+1) &= 2x+2\\
2x+2 &= 2x+2\\
0 &= 0
\end{aligned}
$$

The solution is $x \in \mathbb{R}$**.**

Example 2.3.8 *Solve* $\frac{2}{x+1} = \frac{3}{x+2}$.

Solution:

the common denominator is $(x+1)(x+2)$. Bringing to common denominator we get:

$$\begin{aligned}
\frac{2(x+2)}{(x+1)(x+2)} = \frac{3(x+1)}{(x+1)(x+2)} \\
2(x+2) &= 3(x+1) \\
2x+4 &= 3x+3 \\
2x-3x &= 3-4 \\
-x &= -1 \\
x &= 1
\end{aligned}$$

The solution is $x \in \mathbb{R}$**.**

The equality of fractions can easily be remembered by cross-multiplication: If $\frac{A}{B} = \frac{C}{D}$, by bringing both fractions to the same denominator we get $AD = BC$. This is easily remembered as:

$$\frac{A}{B} = \frac{C}{D}$$

We now solve again Example 2.3.8:

$$\frac{2}{x+1} = \frac{3}{x+2}$$

$$\begin{aligned}
2(x+2) &= 3(x+1) \\
2x+4 &= 3x+3 \\
2x-3x &= 3-4 \\
-x &= -1 \\
x &= 1
\end{aligned}$$

Example 2.3.9 *Solve* $\frac{2x+1}{3x+1} = \frac{4}{7}$.

Solution:

By cross multiplication we get

$$\frac{2x+1}{3x+1} = \frac{4}{7}$$

$$\begin{aligned} 7(2x+1) &= 4(3x+1) \\ 14x+7 &= 12x+4 \\ 14x-12x &= 4-7 \\ 2x &= -3 \\ x &= -\frac{3}{2} \end{aligned}$$

Example 2.3.10 *John's father is 3 times as old as John. In 8 years John's father will be 4 years older than twice John's age. How old is John?*

Solution:

Let x be John's age. His father's age is $3x$.

In 3 years John's age will be $x+8$, while his father age will be $3x+8$. His father age $(3x+8)$ will be 4 more than twice John's age $2(x+8)$. Thus

$$\begin{aligned} 3x+8 &= 2(x+8)+4 \\ 3x+8 &= 2x+16+4 \\ 3x+8 &= 2x+16+4 \\ 3x-2x &= 20-8 \\ x &= 12 \end{aligned}$$

John is 12 years old.

2.3.2 Linear Inequalities

Definition 2.3.3 *An* **inequality** *is a formula of the form $A < B$ or $A \leq B$ or $A > B$ or $A \geq B$, where A, B are expressions containing one or more unknowns.*

By a **solution** *we mean any number or numbers which make the inequality true.*

Example 2.3.11 *Are* $1, 2, 3$ *solutions to* $x^2 - 3 < 0$*?*

Answer:

$$1^2 - 3 < 0\checkmark\,,$$

1 is a solution.

$$2^2 - 3 < 0X\,,$$

2 is not a solution.

$$3^2 - 3 < 0X\,,$$

3 is not a solution.

Important!: By solving an inequality we understand finding **all** solutions.

Important!: Often inequalities have intervals as solutions. Since intervals contain infinitely many points, it follows that often we need to find infinitely many solutions.

To solve linear inequalities we proceed the same way as when we solve equations. There is only one thing we have to be extremely careful with:

When we multiply or divide an inequality by a negative number, the sign switches.

For example

$$-2x < -4\,.$$

If we want to divide by -2 we get

$$x > 2\,.$$

This is because, multiplication by $-$ is the same as switching sides. Indeed, if we move terms on the other side

$$-2x < -4\,.$$

becomes

$$4 < 2x\,.$$

Note that $2x$ is now positive, but faces the inequality from the other side.

Example 2.3.12 *Solve* $2x + 3 < 6$.

Solution :

$$\begin{aligned} 2x + 3 &< 6 \\ 2x &< 6 - 3 \\ 2x &< 3 \\ x &< \frac{3}{2} \end{aligned}$$

Answer: $(-\infty, \frac{3}{2})$.

Example 2.3.13 *Solve* $-(2 + x) < 3(x + 1)$.

Solution :

$$\begin{aligned} -2 - x &< 3x + 3 \\ -x - 3x &< 3 + 2 \\ -4x &< 5 \\ x &> \frac{5}{-4} = -\frac{5}{4} \end{aligned}$$

Answer: $(-\frac{5}{4}, \infty)$.

Example 2.3.14 *Solve* $13x - 12 < 2x + 11$.

Solution :

$$\begin{aligned} 13x - 2x &\leq 11 + 12 \\ 11x &\leq 23 \\ x &\leq \frac{23}{11} \end{aligned}$$

Answer: $[\frac{23}{11}, \infty)$.

Example 2.3.15 *Solve* $2x + 1 \leq 3x - 2 < 5x - 1$.

Solution : Our inequality is made from two inequalities: $2x + 1 \leq 3x - 2$ **and** $3x - 2 < 5x - 1$.

We solve the first inequality:

$$\begin{aligned} 2x + 1 &\leq 3x - 2 \\ 2x - 3x &\leq -2 - 1 \\ -x &\leq -3 \\ x &\geq 3 \end{aligned}$$

The solution to this inequality is $[3, \infty)$.

We now solve the second inequality:

$$\begin{aligned} 3x - 2 &< 5x - 1 \\ 3x - 5x &< -1 + 2 \\ -2x &< 1 \\ x &> -\frac{1}{2} \end{aligned}$$

The solution to this inequality is $(-\frac{1}{2}, \infty)$.

We want x to satisfy BOTH inequalities, which means we intersect the solutions.

Answer: $[3, \infty) \cap (-\frac{1}{2}, \infty) = [3, \infty)$.

Important!: When we solve three or more inequalities in a row, we solve consecutive inequalities and intersect the solutions.

Example 2.3.16 *Solve* $3x + 1 \geq x - 5 \geq 1 + 4x$.

Solution :

We solve the first inequality:

$$\begin{aligned} 3x + 1 &\geq x - 5 \\ 2x &\leq -6 \\ x &\geq -3 \end{aligned}$$

The solution to this inequality is $[-3, \infty)$.

We now solve the second inequality:

$$\begin{aligned} x - 5 &\geq 1 + 4x \\ -3x &\geq 6 \\ x &\leq -2 \end{aligned}$$

The solution to this inequality is $(-\infty, -2]$.

We want x to satisfy BOTH inequalities, which means we intersect the solutions.

Answer: $[-3, \infty) \cap (-\infty, -2] = [-3, -2]$.

Example 2.3.17 *Solve* $3x + 1 \leq 2x + 3 \leq 3x - 1$.

Solution :

We solve the first inequality:

$$\begin{aligned} 3x + 1 &\leq 2x + 3 \\ x &\leq 2 \end{aligned}$$

The solution to this inequality is $(\infty, 2]$.

We now solve the second inequality:

$$\begin{aligned} 2x + 3 &\leq 3x - 1 \\ -x &\leq -4 \\ x &\geq 4 \end{aligned}$$

The solution to this inequality is $[4, \infty)$.

We want x to satisfy BOTH inequalities, which means we intersect the solutions.

Answer: $(\infty, 2] \cap [4, \infty) = \emptyset$.

There is no solution.

Example 2.3.18 *Find the mistake in the following problem:*

We solve

$$x^2 \geq 3x\,.$$

dividing both sides x *we get*

$$x \geq 3\,.$$

Which means that the answer is $[3, \infty)$.

Note that the answer is wrong, as $x = -1$ *is also a solution.*

Solution :

The mistake we did was dividing by x. Note that if x is negative, we have to switch the inequality, but we never took this into account.

2.3.3 Exercises

Exercise 2.3.1 *Solve the following equations:*

i) $3 + \frac{1}{3}x = 5$

ii) $5(2x - 1) = 4(x + 2)$.

iii) $\frac{2}{x+3} = \frac{3}{x+2}$.

Exercise 2.3.2 *Jon thinks about some number* x*. He sees that if he double his number and adds 5 he obtain the same result as when he triples the number and subtracts two. What is Jon's number?*

Exercise 2.3.3 *Solve the following inequalities*

i) $-(3 + x) < 2(3x + 2)$

ii) $3x + 1 < 5x + 2$

Exercise 2.3.4 *Solve the following inequalities*

i) $\frac{x}{2} - \frac{x}{3} \leq \frac{2x+1}{4}$

ii) $\frac{2x+1}{-3} \leq \frac{3x+1}{5}$

Exercise 2.3.5 *Solve*

$$3x + 1 \geq x - 5 \geq 1 + 4x\,.$$

Exercise 2.3.6 *Solve the following equations*

a) $3x + 5 = 5x - 7$.

b) $\frac{2}{x} = \frac{3}{x+2}$.

Exercise 2.3.7 *Solve the following equations*

a) $2x + 5 = 4x - 7$.

b) $\frac{3}{x+1} = \frac{4}{x-2}$.

Exercise 2.3.8 *Solve the following inequalities:*

a) $\frac{3x+1}{4} < \frac{5x+2}{3}$.

b) $2x + 1 \leq 3x + 2 \leq 2x + 7$.

Exercise 2.3.9 *Solve the following inequalities:*

a) $2x + 1 < 4x - 3$.

b) $x + 1 \leq 3x + 5 \leq x + 7$.

2.4 Systems of Linear Equations

By a system of linear equations we understand a group of linear equations. We use an accolade $\{$ to emphasize that the equations are grouped together.

Example 2.4.1

$$\left\{ \begin{array}{lll} x & +2y & = 8 \\ 2x & +2y & = 7 \end{array} \right.\,,$$

and

$$\left\{ \begin{array}{llll} & y & +2z & = 2 \\ x & +3y & +3z & = -2 \\ 2x & -7y & +z & = -2 \end{array} \right.\,.$$

are systems of linear equations.

By a solution to a system we understand a set of numbers which satisfy **all** equations. For example $x = 1, y = 2$ is a solution for

$$\begin{cases} x & +2y & = 5 \\ 2x & +2y & = 6 \end{cases} .$$

Important!: By solving a system we understand finding **all** solutions to that system.

Theorem 2.4.1 *A system of equations has either no solution, one solution, or infinitely many solutions.*

We will show next one example for each of these situations.

Example 2.4.2 *Solve*

$$\begin{cases} x & +2y & = 7 \\ 2x & +2y & = 8 \end{cases} .$$

Solution: Note that the second equation is

$$x + (x + 2y) = 8 .$$

Since $x + 2y = 7$ we get $x + 7 = 8$ therefore $x = 1$.

Subbing in the first equation we get $1 + 2y = 7$. This yields $2y = 6$ and hence $y = 3$. We got

$$\begin{cases} x & = 1 \\ y & = 3 \end{cases} .$$

Important!: This is exactly what we mean by the system having one solution: we get $x =$ a number, $y =$ a number (and if there is a z, $z =$ a number).

Example 2.4.3 *Solve*

$$\begin{cases} x & +y & = 3 \\ 2x & +2y & = 8 \end{cases} .$$

Solution: If we double the first equation we get

$$\begin{cases} 2x & +2y & = 6 \\ 2x & +2y & = 8 \end{cases} .$$

Those two equations contradict eachother, it is impossible to get $2x + 2y$ to be both 6 and 8. This system has no solution.

Example 2.4.4 *Solve*

$$\begin{cases} x & +3y & = 3 \\ 3x & +9y & = 9 \end{cases} .$$

Solution: Note that by multiplying the first equation by 3 we get exactly the second equation. This means that the second equation doesn't tell us anything new, we can ignore it.

We need to solve

$$x + 3y = 3 .$$

There is no way of getting both x and y from here. The best we can do is to find x in terms of y. We get

$$x = 3 - 3y .$$

This means that y can be any real number, and, for each value of y, x is exactly $3 - 3y$. We usually write the answer as $y = t$ and $x = 3 - 3t$ where t is a real number (**parameter**).

$$\begin{cases} x & = 3 - 3t \\ y & = t \end{cases} .$$

Important!: This is exactly what we mean by the system having one solution: we get $x =$ a number, $y =$ a number (and if there is a z, $z =$ a number).

2.4.1 Methods of solving systems of equations

Substitution Method

Pick one equation. Find one variable from this equation. Replace it in the remaining equations.

This way we get a smaller system.

At any point, if we get a false statement, like $3 = 5$, we stop **there is no solution**.

If we get any true statement at any point, like $4 = 4$, we just ignore it.

Last, if we are left with one equation with more variables, find one variable in terms of the other variables, and assign letters (parameters) to each variable which you cannot find at this step.

Reduction Method

Pick two equations. By multiplying each equation by some number, we can make the coefficient of one variable match. By adding or subtracting the equations we can eliminate that variable.

If you have more than 2 equations, pick one of the equations we used with an unused equation, and **eliminate the same variable**.

Repeat as long as you have unused equations.

At any point, if we get a false statement, like $3 = 5$, we stop **there is no solution**.

If we get any true statement at any point, like $4 = 4$, we just ignore it.

Last, if we are left with one equation with more variables, find one variable in terms of the other variables, and assign letters (parameters) to each variable which you cannot find at this step.

2.4.2 Systems of two equations with two unknowns

Example 2.4.5 *Solve*

$$\left\{ \begin{array}{lll} 2x & +3y & = 2 \\ 3x & +4y & = 7 \end{array} \right. .$$

Solution:

Substitution Method From the first equation we get

$$2x = 2 - 3y .$$

hence

$$x = 1 - \frac{3}{2}y .$$

Substitution in the second equation we get

$$
\begin{aligned}
&3\left(1-\frac{3}{2}y\right)+4y=7\\
&3-\frac{9}{2}y+4y=7\\
&-\frac{1}{2}y=4\\
&\frac{-y}{2}=4\\
&-y=8\\
&y=-8
\end{aligned}
$$

Therefore

$$x=1-\frac{3}{2}(-8)=1-(-12)=13\,.$$

The answer is

$$\begin{cases} x &= 13\\ y &= -8\end{cases}.$$

Reduction Method

Multiplying the first equation by 3 and the second by 2 we get

$$\begin{cases} 6x & +9y & = 6\\ 6x & +8y & = 14\end{cases}.$$

Subtracting we get

$$y=-8\,.$$

Now, subbing in the first equation we get

$$2x-24=2\,.$$

Therefore

$$2x=26 \Rightarrow x=13\,.$$

Thus

$$\begin{cases} x &= 13\\ y &= -8\end{cases}.$$

Example 2.4.6 *Solve*

$$\begin{cases} x & +2y & = 2 \\ 3x & +6y & = 9 \end{cases}.$$

Solution:

Substitution Method From the first equation we get

$$x = 2 - 2y\,.$$

Substitution in the second equation we get

$$\begin{aligned} 3\,(2 - 2y) + 6y &= 9 \\ 6 - 6y + 6y &= 9 \\ 6 &= 9 \end{aligned}$$

As this statement is false **This equation has no solutions**.

Reduction Method

Multiplying the first equation by 3 get

$$\begin{cases} 3x & +6y & = 6 \\ 3x & +6y & = 9 \end{cases}.$$

Subtracting we get

$$0 = -3\,.$$

As this statement is false **This equation has no solutions**.

Example 2.4.7 *Solve*

$$\begin{cases} x & -2y & = 3 \\ -2x & +4y & = -6 \end{cases}.$$

Solution:

Substitution Method From the first equation we get

$$x = 3 + 2y\,.$$

Substitution in the second equation we get

$$
\begin{aligned}
-2\,(3+2y)+4y &= -6\\
-6-4y+4y &= -6\\
-6 &= -6
\end{aligned}
$$

As this statement is **true** we can ignore the second equation. We need to solve

$$x-2y=3\,.$$

Then y moves on the other side and becomes a parameter.

$$x=2y+3\,.$$

As y is a parameter, we assign a letter $y=t$. Then $x=2t+3$.

The answer is

$$\begin{cases} x &= 2t+3\\ y &= t \end{cases}\,.$$

Reduction Method

Multiplying the first equation by 2 we get

$$\begin{cases} 2x & -4y & = 6\\ -2x & +4y & = -6 \end{cases}\,.$$

Adding we get

$$0=0\,.$$

We can ignore the second equation. We need to solve

$$x-2y=3\,.$$

Then y moves on the other side and becomes a parameter.

$$x=2y+3\,.$$

As y is a parameter, we assign a letter $y=t$. Then $x=2t+3$.

The answer is

$$\begin{cases} x &= 2t+3\\ y &= t \end{cases}\,.$$

Example 2.4.8 *Solve*

$$\begin{cases} 2x & +4y & = 3 \\ 3x & +3y & = 4 \end{cases} .$$

Solution:

Substitution Method From the first equation we get

$$2x = 3 - 4y \, .$$

hence

$$x = \frac{3}{2} - 2y \, .$$

Substitution in the second equation we get

$$\begin{aligned} &3\left(\frac{3}{2} - 2y\right) + 3y = 4 \\ &\frac{9}{2} - 6y + 3y = 4 \\ &\quad - 3y = 4 - \frac{9}{2} \\ &\quad - 3y = -\frac{1}{2} \\ &\quad y = \frac{1}{6} \end{aligned}$$

Therefore

$$x = \frac{3}{2} - 2 \cdot \frac{1}{6} = \frac{7}{6} \, .$$

The answer is

$$\begin{cases} x & = \frac{7}{6} \\ y & = \frac{1}{6} \end{cases} .$$

Reduction Method

Multiplying the first equation by 3 and the second by 2 we get

$$\begin{cases} 6x & +12y & = 9 \\ 6x & +6y & = 8 \end{cases} .$$

Subtracting we get

$$6y = 1 \Rightarrow y = \frac{1}{6} \, .$$

Now, subbing in the first equation we get

$$2x + 4 \cdot \frac{1}{6} = 3\,.$$

Therefore

$$2x = 3 - \frac{4}{6} = \frac{14}{6}\,.$$

Thus

$$\begin{cases} x &= \frac{7}{6} \\ y &= \frac{1}{6} \end{cases} .$$

2.4.3 Systems with 3 unknowns

Example 2.4.9 *Solve*

$$\begin{cases} 3x & +5y & +3z & = 2 \\ 3x & +2y & +z & = 7 \\ 6x & +7y & +6z & = 12 \end{cases} .$$

Solution:

Substitution Method From the second equation we get

$$z = 7 - 3x - 2y\,.$$

Substitution in the first equation we get

$$3x + 5y + 3(7 - 3x - 2y) = 2$$
$$3x + 5y + 21 - 9x - 6y = 2$$
$$-6x - y = -19$$
$$6x + y = 19$$

Substitution in the third equation we get

$$6x + 7y + 6(7 - 3x - 2y) = 12$$
$$6x + 7y + 42 - 18x - 12y = 12$$
$$-12x - 5y = -30$$
$$12x + 5y = 30$$

Therefore we get the smaller system:

$$\begin{cases} 6x & +y & = 19 \\ 12x & +5y & = 30 \end{cases} .$$

From the first equation we get

$$y = 19 - 6x .$$

Plugging into the second we get

$$\begin{aligned} 12x + 5(19 - 6x) &= 30 \\ 12x + 95 - 30x &= 30 \\ -18x &= -65 \\ x &= \frac{65}{18} \end{aligned}$$

Then

$$y = 19 - 6\frac{65}{18} = 19 - \frac{65}{3} = \frac{57 - 65}{3} = -\frac{8}{3} .$$

and

$$z = 7 - 3\frac{65}{18} - 2(-\frac{8}{3}) = 7 - \frac{65}{6} + \frac{16}{3} = \frac{42 - 65 + 32}{6} = \frac{9}{6} = \frac{3}{2} .$$

The answer is

$$\begin{cases} x & = \frac{65}{18} \\ y & = -\frac{8}{3} \\ z & = \frac{3}{2} \end{cases} .$$

Reduction Method

We group the first two equations:

$$\begin{cases} 3x & +5y & +3z & = 2 \\ 3x & +2y & +z & = 7 \end{cases} .$$

Subtracting we get

$$3y + 2z = -5 .$$

We now group first and third equation

$$\begin{cases} 3x & +5y & +3z & = 2 \\ 6x & +7y & +6z & = 12 \end{cases} .$$

Doubling the first equation we get

$$\begin{cases} 6x & +10y & +6z & = 4 \\ 6x & +7y & +6z & = 12 \end{cases}.$$

Subtracting we get

$$3y = -8\,.$$

Thus our system is

$$\begin{cases} 3y & +2z & = -5 \\ 3y & & = -8 \end{cases}.$$

From here we get $y = -\frac{8}{3}$ and

$$2z = -5 - 3y = -5 + 8 = 3\,.$$

Thus

$$z = \frac{3}{2}\,.$$

The first equation now gives

$$\begin{aligned} 3x + 5(-\frac{8}{3}) + 3 \cdot \frac{3}{2} &= 2 \\ 3x - \frac{40}{3} + \frac{9}{2} &= 2 \\ 3x &= \frac{40}{3} - \frac{9}{2} + 2 \\ 3x &= \frac{80 - 27 + 12}{6} \\ 3x &= \frac{65}{6} \\ x &= \frac{65}{18} \end{aligned}$$

The answer is

$$\begin{cases} x & = \frac{65}{18} \\ y & = -\frac{8}{3} \\ z & = \frac{3}{2} \end{cases}.$$

Important!: Systems often have solutions with fractions.

Important!: When we solve a system one method is enough. We can also combine the two methods: start solving the system by the substitution method, and when we get to the smaller system use the reduction method to solve this one.

Example 2.4.10 *Solve*

$$\begin{cases} x & +2y & +3z & = 2 \\ 2x & +3y & +z & = 7 \\ 4x & +7y & +8z & = 12 \end{cases} .$$

Solution:

Substitution Method From the first equation we get

$$x = 2 - 2y - 3z\,.$$

Substitution in the second equation we get

$$\begin{aligned} 2(2-2y-3z)+3y+z = 74 \quad & -4y-6z+3y+z = 7 \\ -y-5z &= 3 \\ y+5z &= -3 \end{aligned}$$

Substitution in the third equation we get

$$\begin{aligned} 4(2-2y-3z)+7y+8z = 128 \quad & -8y-12z+7y+8z = 12 \\ -y-4z &= 4 \\ y+4z &= -4 \end{aligned}$$

Therefore we get the smaller system:

$$\begin{cases} y & +5z & = -3 \\ y & +4z & = -4 \end{cases} .$$

Subtracting the two equations we get $z = 1$. Hence

$$y + 5 = -3 \Rightarrow y = -8\,.$$

Therefore

$$x = 2 - 2(-8) - 3 \cdot 1 = 2 + 16 - 3 = 15\,.$$

The answer is

$$\begin{cases} x & = 15 \\ y & = -8 \\ z & = 1 \end{cases} .$$

Reduction Method

We group the first two equations:

$$\begin{cases} x & +2y & +3z & = 2 \\ 2x & +3y & +z & = 7 \end{cases} .$$

Doubling the first equation we get

$$\begin{cases} 2x & +4y & +6z & = 4 \\ 2x & +3y & +z & = 7 \end{cases} .$$

Subtracting we get

$$y + 5z = -3 \, .$$

We now group last two equations

$$\begin{cases} 2x & +3y & +z & = 7 \\ 4x & +7y & +8z & = 12 \end{cases} .$$

Doubling the first equation we get

$$\begin{cases} 4x & +6y & +2z & = 14 \\ 4x & +7y & +8z & = 12 \end{cases} .$$

Subtracting we get

$$-y - 6z = 2 \, .$$

or

$$y + 6z = -2 \, .$$

Thus our system is

$$\begin{cases} y & +5z & = -3 \\ y & +6z & = -2 \end{cases} .$$

Subtracting, we get

$$-z = -1 \, .$$

Thus

$$z = 1\,.$$

Then

$$y + 6 = -2 \Rightarrow y = -8\,.$$

The first equation now gives

$$\begin{aligned} &x + 2(-8) + 3 = 2x \quad = 2 + 16 - 3 \\ &x = 15 \end{aligned}$$

The answer is

$$\begin{cases} x &= 15 \\ y &= -8 \\ z &= 1 \end{cases}.$$

Example 2.4.11 *Solve*

$$\begin{cases} x & +y & +2z & = 2 \\ 2x & +3y & +z & = 7 \\ 3x & +4y & +3z & = 9 \end{cases}.$$

Solution:

Substitution Method From the first equation we get

$$x = 2 - y - 2z\,.$$

Substitution in the second equation we get

$$\begin{aligned} &2(2 - y - 2z) + 3y + z = 7 \\ &4 - 2y - 4z + 3y + z = 7 \\ &y - 3z = 3 \end{aligned}$$

Substitution in the third equation we get

$$\begin{aligned} &3(2 - y - 2z) + 4y + 3z = 9 \\ &6 - 3y - 6z + 4y + 3z = 9 \\ &y - 3z = 3 \end{aligned}$$

We got the same equation twice, thus we can ignore the second one. The system is reduced to

$$y - 3z = 3\,.$$

As we only have one equation with 2 unknowns, one of the variables becomes a parameter.

$$z = t\,.$$

Then

$$\begin{aligned} y &= 3 + 3t \\ x &= 2 - (3 + 3t) - 2t = -1 - 5t\,. \end{aligned}$$

The answer is

$$\begin{cases} x &= -1 - 5t \\ y &= 3 + 3t \\ z &= t \end{cases}.$$

Reduction Method

We group the first two equations:

$$\begin{cases} x & +y & +2z & = 2 \\ 2x & +3y & +z & = 7 \end{cases}.$$

Doubling the first equation we get

$$\begin{cases} 2x & +2y & +4z & = 4 \\ 2x & +3y & +z & = 7 \end{cases}.$$

Subtracting we get

$$-y + 3z = -3\,.$$

We now group last two equations

$$\begin{cases} 2x & +3y & +z & = 7 \\ 3x & +4y & +3z & = 9 \end{cases}.$$

Multiplying the first equation by 3 and second by 2 we get

$$\begin{cases} 6x & +9y & +3z & = 21 \\ 6x & +8y & +6z & = 18 \end{cases}.$$

Subtracting we get

$$y - 3z = 3\,.$$

Thus our system is

$$\begin{cases} -y & +3z & = -3 \\ y & -3z & = 3 \end{cases}.$$

Adding the two equations we get $0 = 0$, which is telling us that we can ignore one equation. Thus, we need to solve

$$y - 3z = 3\,.$$

As we only have one equation with 2 unknowns, one of the variables becomes a parameter.

$$z = t\,.$$

Then

$$y = 3 + 3t\,.$$

Plugging in the first equation we get

$$x + (3 + 3t) + 2t = 2\,.$$

Thus the answer is

$$\begin{cases} x & = -1 - 5t \\ y & = 3 + 3t \\ z & = t \end{cases}.$$

Example 2.4.12 *Solve*

$$\begin{cases} x & +3y & +2z & = 7 \\ 2x & -4y & +z & = 2 \\ 3x & -y & +3z & = 10 \end{cases}.$$

Solution:

Substitution Method From the first equation we get

$$x = 7 - 3y - 2z\,.$$

Substitution in the second equation we get

$$2(7-3y-2z)-4y+z=2$$
$$14-6y-4z-4y+z=2$$
$$-10y-3z=-12$$
$$10y+3z=12$$

Substitution in the third equation we get

$$3(7-3y-2z)-y+3z=1021 \quad -9y-6z-y+3z=10$$
$$-10y-3z=-11$$
$$10y+3z=11$$

The system we get is

$$\left\{\begin{array}{lll} 10y & +3z & =12 \\ 10y & +3z & =11 \end{array}\right. .$$

Subtracting we get

$$0=1\,.$$

This means that **This system has no solution**.

Reduction Method

We group the first two equations:

$$\left\{\begin{array}{llll} x & +3y & +2z & =7 \\ 2x & -4y & +z & =2 \end{array}\right. .$$

Doubling the first equation we get

$$\left\{\begin{array}{llll} 2x & +6y & +4z & =14 \\ 2x & -4y & +z & =2 \end{array}\right. .$$

Subtracting we get

$$10y+3z=12\,.$$

We now group last two equations

$$\left\{\begin{array}{llll} 2x & -4y & +z & =2 \\ 3x & -y & +3z & =10 \end{array}\right. .$$

Multiplying the first equation by 3 and second by 2 we get

$$\begin{cases} 6x & -12y & +3z & = 6 \\ 6x & -2y & +6z & = 10 \end{cases}.$$

Subtracting we get

$$-10y - 3z = -4\,.$$

Thus our system is

$$\begin{cases} 10y & +3z & = 12 \\ -10y & -3z & = -4 \end{cases}.$$

Adding the two equations we get $0 = 8$, **This system has no solution**.

Example 2.4.13 *Solve*

$$\begin{cases} x & -2y & +z & = 2 \\ -2x & +4y & -2z & = -4 \\ 3x & -6y & +3z & = 6 \end{cases}.$$

Solution:

Substitution Method From the first equation we get

$$x = 2 + 2y - z\,.$$

Substitution in the second equation we get

$$\begin{aligned} &-2(2 + 2y - z) + 4y - 2z = -4 \\ &-4 - 4y + 2z + 4y - 2z = -4 \\ &-4 = -4 \end{aligned}$$

As this is true, **we should ignore the second equation**.
Substitution in the third equation we get

$$\begin{aligned} &3(2 + 2y - z) - 6y + 3z = 66 + 6y - 3z - 6y + 3z = 6 \\ &6 = 6 \end{aligned}$$

As this is true, **we should ignore the third equation**.

Therefore, wee need to solve

$$x - 2y + z = 2\,.$$

As we only have one equation, two unknowns become parameters. $y = t, z = s$.

Then

$$x = 2 + 2t - s\,.$$

The answer is

$$\begin{cases} x &= 2 + 2t - s \\ y &= t \\ z &= s \end{cases}\,.$$

Reduction Method

We group the first two equations:

$$\begin{cases} x & -2y & +z & = 2 \\ -2x & +4y & -2z & = -4 \end{cases}\,.$$

Doubling the first equation we get

$$\begin{cases} 2x & -4y & +2z & = 4 \\ -2x & +4y & -2z & = -4 \end{cases}\,.$$

Adding we get

$$0 = 0\,.$$

We should ignore the second equation.

We now group last two equations

$$\begin{cases} -2x & +4y & -2z & = -4 \\ 3x & -6y & +3z & = 6 \end{cases}\,.$$

Multiplying the first equation by 3 and second by 2 we get

$$\begin{cases} -6x & +12y & -6z & = -12 \\ 6x & -12y & +6z & = 12 \end{cases}\,.$$

Adding we get

$$0 = 0\,.$$

We should ignore the third equation.

Therefore, wee need to solve

$$x - 2y + z = 2\,.$$

As we only have one equation, two unknowns become parameters. $y = t, z = s$.

Then

$$x = 2 + 2t - s\,.$$

The answer is

$$\begin{cases} x &= 2 + 2t - s \\ y &= t \\ z &= s \end{cases}\,.$$

2.4.4 Exercises

Exercise 2.4.1 *Solve the following systems of equations*

i)

$$\begin{cases} 2x + 3y = 5 \\ 5x + 7y = 8 \end{cases}\,.$$

ii)

$$\begin{cases} 3x - y = 5 \\ 4x - 2y = 8 \end{cases}\,.$$

Exercise 2.4.2 *Solve the following systems of equations*

i)

$$\begin{cases} 2x + y = 5 \\ 4x + 2y = 8 \end{cases}\,.$$

ii)

$$\begin{cases} 2x - y = 4 \\ 4x - 2y = 8 \end{cases}\,.$$

Exercise 2.4.3 *For which real number a does the following system have solutions?*

$$\begin{cases} x + 2y = 4 \\ 2x + 4y = a \end{cases}.$$

Exercise 2.4.4 *Solve the following systems of equations*

i)

$$\begin{cases} x & & -3z & = 8 \\ 2x & +2y & +9z & = 7 \\ & y & +5z & = -2 \end{cases}.$$

ii)

$$\begin{cases} & y & +5z & = -4 \\ x & +4y & +3z & = -2 \\ 2x & +7y & +z & = -2 \end{cases}.$$

Exercise 2.4.5 *Solve the following systems of equations*

i)

$$\begin{cases} x & -2y & +z & = 0 \\ & 2y & -8z & = 8 \\ -4x & +5y & +9z & = -9 \end{cases}.$$

ii)

$$\begin{cases} x & +5y & & = 21 \\ & y & +z & = 4 \\ 2x & +11y & +z & = 46 \end{cases}.$$

Exercise 2.4.6 *Solve the following systems of equations:*

a)

$$\begin{cases} 3x + 2y = 4 \\ 6x + 4y = 4 \end{cases}.$$

b)

$$\begin{cases} 2x + 3y + 7z = 5 \\ 5x + 7y + 2z = 8 \\ 3x + 4y - 4z = 2 \end{cases}.$$

Exercise 2.4.7 *Solve the following systems of equations:*

a)

$$\begin{cases} x+3y=4 \\ 2x+7y=9 \end{cases}.$$

b)

$$\begin{cases} x+2y-3z=2 \\ 2x+3y+z=1 \\ 3x+5y-2z=3 \end{cases}.$$

Exercise 2.4.8 *Solve*

$$\begin{cases} x+y-2z=0 \\ 2x+3y-z=4 \\ 3x+4y-2z=3 \end{cases}.$$

Exercise 2.4.9 *Solve*

$$\begin{cases} x+y-2z=0 \\ 2x+3y-z=4 \\ 5x+7y-3z=9 \end{cases}.$$

2.5 Completing the square

By a **quadratic expression** we understand an expression of the form ax^2+bx+c, where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

Often, the term bx creates problems. Completing the square is a simple way of elimination this term.

The idea is to use backwards the formula

$$(x+a)^2 = x^2+2ax+a^2 .$$

Completing the square: Whenever when we see $x^2 \pm ax$ we can complete the square by adding and subtraction $(\frac{a}{2})^2$. This way we get

$$x^2 \pm ax = x^2 \pm ax + (\frac{a}{2})^2 - (\frac{a}{2})^2 = \left(x \pm \frac{a}{2}\right)^2 - (\frac{a}{2})^2 .$$

This way we completed the square $\left(x \pm \frac{a}{2}\right)^2$.

This is easy to remember, you take the coefficient of x, half it and square.

Example 2.5.1 *Complete the square in*

$$x^2 + 2x\,.$$

Solution:

$$x^2 + 2x = x^2 + 2x + 1^2 - 1^2 = (x+1)^2 - 1\,.$$

Example 2.5.2 *Complete the square in*

$$x^2 + 6x\,.$$

Solution:

$$x^2 + 6x = x^2 + 6x + 3^2 - 3^2 = (x+3)^2 - 9\,.$$

Example 2.5.3 *Complete the square in*

$$x^2 - 3x\,.$$

Solution:

$$x^2 - 3x = x^2 - 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2 = (x - \frac{3}{2})^2 - \frac{9}{4}\,.$$

Example 2.5.4 *Complete the square in*

$$x^2 + 4x + 5\,.$$

Solution:

$$x^2 + 4x + 5 = x^2 + 4x + 2^2 - 2^2 + 5 = (x+2)^2 + 1\,.$$

Example 2.5.5 *Complete the square in*

$$x^2 + 7x - 2\,.$$

Solution:

$$\begin{aligned} x^2 + 7x - 2 &= x^2 + 7x + (\frac{7}{2})^2 - (\frac{7}{2})^2 - 2 = (x - \frac{7}{2})^2 - \frac{49}{4} - 2 \\ &= (x - \frac{7}{2})^2 - \frac{49}{4} - \frac{8}{4} = (x - \frac{7}{2})^2 - \frac{57}{4} \end{aligned}$$

Example 2.5.6 *Complete the square in*

$$x^2 + \frac{1}{3}x + 2\,.$$

Solution:

$$\begin{aligned} x^2\frac{1}{3}x + 2 &= x^2 + \frac{1}{3}x + (\frac{1}{6})^2 - (\frac{1}{6})^2 + 2 = (x + \frac{1}{6})^2 - \frac{1}{36} + 2 \\ &= (x + \frac{1}{6})^2 - \frac{1}{36} + \frac{72}{36} = (x + \frac{1}{6})^2 + \frac{71}{36} \end{aligned}$$

Example 2.5.7 *Complete the square in*

$$2x^2 + 3x + 1\,.$$

Solution: To complete the square, we need to start with x^2. To do this, we make 2 a common factor by force: $3 = 2 \cdot \frac{3}{2}$ and $1 = 2 \cdot \frac{1}{2}$. Therefore

$$\begin{aligned} 2x^2 + 3x + 1 &= 2x^2 + 2 \cdot \frac{3}{2}x + 2(\frac{1}{2}) = 2\left(x^2 + \frac{3}{2}x + \frac{1}{2}\right) = 2\left(x^2 + \frac{3}{2}x + (\frac{3}{4})^2 - (\frac{3}{4})^2 + \frac{1}{2}\right) \\ &= 2\left(\left(x + \frac{3}{4}\right)^2 - \frac{9}{16} + \frac{1}{2}\right) = 2\left(\left(x + \frac{3}{4}\right)^2 - \frac{9}{16} + \frac{8}{16}\right) \\ &= 2\left(\left(x + \frac{3}{4}\right)^2 - \frac{1}{16}\right) = 2\left(x + \frac{3}{4}\right)^2 - \frac{2}{16} \\ &= 2\left(x + \frac{3}{4}\right)^2 - \frac{2}{16} \end{aligned}$$

2.5.1 Exercises

Exercise 2.5.1 *Complete the square*

i) $x^2 - 10x$.

ii) $x^2 + 2x + 3$.

iii) $2x^2 + 4x - 2$.

Exercise 2.5.2 *Complete the square for*

$$x^2 + 3x + 1\,.$$

Exercise 2.5.3 *Complete the square for*

a) $x^2 + 4x - 2$.

b) $2x^2 + 3x$.

2.6 Quadratic Equations

2.6.1 Solving quadratic equations by completing the square

Definition 2.6.1 *A* **quadratic equation** *is an equation of the form*

$$ax^2 + bx + c = 0\,.$$

with $a, b, c \in \mathbb{R}$ *and* $a \neq 0$.

Exactly as before, we seek all solutions.

Important!: If we start by dividing by a, and then complete the square, any quadratic equation becomes one of the form

$$(x + e)^2 = f\,.$$

for some real numbers e, f.

If $f < 0$ the left hand side is positive, while the right hand side is negative. Therefore **there is no solution**.

If $f = 0$ then $x + e$ must be zero. Therefore **there is one solution**.

If $f > 0$ then $x + e$ must be either $\sqrt{f}$ or $-\sqrt{f}$. Therefore, in this case **there are two solutions**.

We also get

Theorem 2.6.1 *A quadratic equation has 0,1 or 2 solutions.*

Example 2.6.1 *Solve*

$$x^2 + 4x - 5 = 0\,.$$

Solution:

$$\begin{aligned} x^2 + 4x + 2^2 - 2^2 - 5 &= 0 \\ (x+2)^2 - 9 &= 0 \\ (x+2)^2 &= 9x + 2 \quad = \pm 3 \end{aligned}$$

If $x + 2 = 3$ we get $x = 1$, while if $x + 2 = -3$ we get $x = -5$.
Answer: $x = 1, x = -5$.

Example 2.6.2 *Solve*

$$x^2 - 6x + 11 = 0\,.$$

Solution:

$$\begin{aligned} x^2 - 6x + 3^2 - 3^2 + 11 &= 0 \\ (x-3)^2 + 2 &= 0 \\ (x-3)^2 &= -2 \end{aligned}$$

Answer: No solution.

Example 2.6.3 *Solve*

$$x^2 - 3x - 7 = 0\,.$$

Solution:

$$\begin{aligned} x^2 - 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2 - 7 &= 0 \\ (x - \frac{3}{2})^2 - \frac{9}{4} - 7 &= 0 \\ (x - \frac{3}{2})^2 &= \frac{9}{4} + 7 \\ (x - \frac{3}{2})^2 &= \frac{37}{4} \\ x - \frac{3}{2} &= \pm\sqrt{\frac{37}{4}} \\ x - \frac{3}{2} &= \pm\frac{\sqrt{37}}{2} \\ x &= \frac{3}{2} + \pm\frac{\sqrt{37}}{2} \end{aligned}$$

Answer: $x = \frac{3+\sqrt{37}}{2}$ and $x = \frac{3-\sqrt{37}}{2}$.

Example 2.6.4 *Solve*

$$x^2 + 6x + 9 = 0\,.$$

Solution:

$$\begin{aligned} x^2 + 6x + 3^2 - 3^2 + 9 &= 0 \\ (x+3)^2 - 9 + 9 &= 0 \\ (x+3)^2 &= 0 \\ x + 3 &= 0 \\ x &= -3 \end{aligned}$$

Answer: $x = -3$.

Example 2.6.5 *Solve*

$$3x^2 - 5x + 2 = 0\,.$$

Solution: We start by dividing by 3:

$$\begin{aligned} x^2 - \frac{5}{3}x + \frac{2}{3} &= 0 \\ x^2 - \frac{5}{3}x + (\frac{5}{6})^2 - (\frac{5}{6})^2 + \frac{2}{3} &= 0 \\ (x + \frac{5}{6})^2 - \frac{25}{36} + \frac{2}{3} &= 0 \\ (x + \frac{5}{6})^2 &= \frac{25}{36} - \frac{2}{3} \\ (x + \frac{5}{6})^2 &= \frac{1}{36} \\ x + \frac{5}{6} &= \pm\sqrt{\frac{1}{36}} \\ x + \frac{5}{6} &= \pm\frac{1}{6} \\ x &= -\frac{5}{6} \pm \frac{1}{6} \end{aligned}$$

Therefore $x = -\frac{5}{6} + \frac{1}{6} = -\frac{4}{6} = -\frac{2}{3}$ and $x = -\frac{5}{6} - \frac{1}{6} = -\frac{6}{6} = -1$.
Answer: $x = -1$ and $x = -\frac{2}{3}$.

2.6.2 The Quadratic Formula

Consider the quadratic equation

$$ax^2 + bx + c = 0\,.$$

As $a \neq 0$ we can divide by a. We get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0\,.$$

Next lets complete the square.

$$\begin{aligned}
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a} &= 0 \\
\left(x + \frac{b}{2a}\right)^2 - (\frac{b}{2a})^2 + \frac{c}{a} &= 0 \\
\left(x + \frac{b}{2a}\right)^2 &= (\frac{b}{2a})^2 - \frac{c}{a} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{4ac}{2a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2}
\end{aligned}$$

Therefore, we get

- If $b^2 - 4ac < 0$ the equation has no solution.
- If $b^2 - 4ac = 0$ the equation has one solution. This is given by $x + \frac{b}{2a} = 0$, therefore $x = -\frac{b}{2a}$.
- If $b^2 - 4ac > 0$ the equation has two solutions given by $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} = \frac{\sqrt{b^2-4ac}}{2a}$. Therefore

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,.$$

Since the expression $b^2 - 4ac$ plays an important role in solving the quadratic equation, we denote it by a letter.

Notation:

$$\Delta = b^2 - 4ac \, .$$

Thus we get the following Theorem.

Theorem 2.6.2 *Consider the quadratic equation*

$$ax^2 + bx + c = 0 \, ,$$

and let $\Delta = b^2 - 4ac$. *Then*

- *i) If* $\Delta < 0$ *the equation has no solution.*
- *ii) If* $\Delta = 0$ *the equation has one solutio given by*

$$x = -\frac{b}{2a} \, .$$

- *iii) If* $\Delta > 0$ *the equation has two solutions given by*

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} \, .$$

Example 2.6.6 *Solve*

$$x^2 - 3x + 2 = 0 \, .$$

Solution: $a = 1, b = -3, c = 2$. Then $\Delta = (-3)^2 - 4 \cdot 1 \cdot 2 = 9 - 8 = 1$.

Since $\Delta > 0$ this equation has two solutions given by

$$x_{1,2} = \frac{-(-3) \pm \sqrt{1}}{2} = \frac{3 \pm 1}{2} \, .$$

The two solutions are

$$x_1 = \frac{3+1}{2} = 2 \, ; \; x_2 = \frac{3-1}{2} = 1 \, .$$

Example 2.6.7 *Solve*

$$x^2 - 6x + 9 = 0 \, .$$

Solution: $a = 1, b = -6, c = 9$. Then $\Delta = (-6)^2 - 4 \cdot 1 \cdot 9 = 36 - 36 = 0$.

Since $\Delta = 0$ this equation has one solutions given by

$$x = \frac{-(-6)}{2} = 3\,.$$

Example 2.6.8 *Solve*

$$3x^2 + 4x + 10 = 0\,.$$

Solution: $a = 1, b = 4, c = 10$. Then $\Delta = (4)^2 - 4 \cdot 1 \cdot 10 = 16 - 40 < 0$.

This equation has **no solution**.

Example 2.6.9 *Solve*

$$2x^2 + 5x + 3 = 0\,.$$

Solution: $a = 2, b = 5, c = 3$. Then $\Delta = 5^2 - 4 \cdot 2 \cdot 3 = 25 - 24 = 1$.

Since $\Delta > 0$ this equation has two solutions given by

$$x_{1,2} = \frac{-5 \pm \sqrt{1}}{2 \cdot 2} = \frac{-5 \pm 1}{4}\,.$$

The two solutions are

$$x_1 = \frac{-5+1}{4} = -1\,;\; x_2 = \frac{-5-1}{4} = -\frac{3}{2}\,.$$

Example 2.6.10 *Solve*

$$x^2 + x - 1 = 0\,.$$

Solution: $a = 1, b = 1, c = -1$. Then $\Delta = 1^1 - 4 \cdot 1 \cdot (-1) = 1 + 4 = 5$.

Since $\Delta > 0$ this equation has two solutions given by

$$x_{1,2} = \frac{-1 \pm \sqrt{5}}{2} = \,.$$

The two solutions are

$$x_1 = \frac{-1+\sqrt{5}}{2}\,;\; x_2 = \frac{-1-\sqrt{5}}{2}\,.$$

2.6.3 Factoring Quadratics

Note that

$$(X-2)(X-3) = X^2 - 2X - 3X + 6 = X^2 - 5X + 6\,.$$

By factorisation we understand the reverse process: Given a quadratic like $X^2 - 5x + 6$ try to write it in the form $(X-2)(X-3)$.

Example 2.6.11 *Factor*

$$X^2 - 6X + 8\,.$$

Solution: Lets try to work backwards. Lets say that we managed to factor the expression. Then we get some r, s so that

$$X^2 - 6X + 8 = (X-r)(X-s)\,.$$

Foiling we get

$$X^2 - 6X + 8 = X^2 - rX - sX + rs = X^2 - (r+s)X + rs\,.$$

Therefore, the r, s we found satisfy

$$\begin{cases} r+s &= 6 \\ rs &= 8 \end{cases}\,.$$

It is easy now to guess that the right values are 2 and 4. Therefore, the answer is

$$X^2 - 6X + 8 = (X-4)(X-2)\,.$$

Important! If the factorization is easy to guess, then solving the quadratic is very easy. Indeed, if we know that

$$X^2 - 6X + 8 = (X-4)(X-2)\,.$$

to solve

$$X^2 - 6X + 8 = 0\,.$$

we can solve instead

$$(X-4)(X-2) = 0\,.$$

The only way the product be zero is if $X - 4 = 0$ or $X - 2 = 0$.

Theorem 2.6.3 *If*

$$X^2 + bX + c = (X - r)(X - s)\,,$$

then

- *i)* $r + s = -b$.
- *ii)* $rs = c$.
- *iii)* r, s *are the solutions to* $X^2 + bX + c = 0$.

Important! The Theorem tells us the following. To find the factorization of X^2+bX+C we start by seeking two numbers r, s so that $r+s = -b, rs = c$. If they are easy to find, we just guess them. If they are not easy to find/guess, we simply solve instead the quadratic

$$X^2 + bX + c = 0\,.$$

If this equation has two solutions, those are r and s. If the equation has one solution, then $r = s$ is the solution. if the quadratic has no solution, it means the quadratic cannot be factored.

Example 2.6.12 *Factor*

$$X^2 - 4X + 3\,.$$

Solution: We seek r, s so that

$$\begin{cases} r+s & = 4 \\ rs & = 3 \end{cases}\,.$$

It is easy now to guess that the right values are 1 and 3. Therefore, the answer is

$$X^2 - 4X + 3 = (X - 1)(X - 3)\,.$$

Example 2.6.13 *Factor*

$$X^2 + 7X + 10\,.$$

Solution: We seek r, s so that

$$\begin{cases} r+s &= -7 \\ rs &= 10 \end{cases} .$$

It is easy now to guess that the right values are -2 and -5. Therefore, the answer is

$$X^2 + 7X + 10 = (X - (-2))(X - (-5)) = (X + 2)(X + 5) .$$

Example 2.6.14 *Factor*

$$X^2 + 6X + 9 .$$

Solution: We seek r, s so that

$$\begin{cases} r+s &= -6 \\ rs &= 9 \end{cases} .$$

It is easy now to guess that the right values are -3 and -3. Therefore, the answer is

$$X^2 + 6X + 9 = (X - (-3))(X - (-3)) = (X + 3)(X + 3) .$$

Example 2.6.15 *Factor*

$$X^2 + 3X + 6 .$$

Solution: We seek r, s so that

$$\begin{cases} r+s &= -3 \\ rs &= 6 \end{cases} .$$

As we cannot guess the numbers, we solve the equation

$$X^2 + 3X + 6 = 0 .$$

$$\Delta = 9 - 24 < 0 .$$

This equation has no solution, which means that

$$X^2 + 3X + 6$$

cannot be factored.

Example 2.6.16 *Factor*

$$X^2 + 2X - 5\,.$$

Solution: We seek r, s so that

$$\begin{cases} r+s &= -2 \\ rs &= -5 \end{cases}.$$

As we cannot guess the numbers, we solve the equation

$$X^2 + 2X - 5 = 0\,.$$

$$\Delta = 4 + 20 = 24\,.$$

Therefore

$$x_{1,2} = \frac{-2 \pm \sqrt{24}}{2} = \frac{-2 \pm 2\sqrt{6}}{2} = -1 \pm \sqrt{6}\,.$$

The two roots are $-1-\sqrt{6}$ and $-1+\sqrt{6}$. These are our r and s.

This tells us that

$$X^2+2X-5 = \left(X - (-1-\sqrt{6})\right)\left(X - (-1+\sqrt{6})\right) = \left(X+1+\sqrt{6}\right)\left(X+1-\sqrt{6}\right)\,.$$

Question 2.6.1 *What if we want to factor something like*

$$aX^2 + bX + c\,.$$

Theorem 2.6.4 *If it can be factored, then*

$$aX^2 + bX + c = a(X-r)(X-s)\,,$$

Where

- *i)* $r + s = -\frac{b}{a}$.
- *ii)* $rs = \frac{c}{a}$.
- *iii)* r, s *are the solutions to* $aX^2 + bX + c = 0$.

Important! The Theorem tells us the following. To find the factorization of aX^2+bX+C we start by seeking two numbers r,s so that $r+s=-\frac{b}{a}, rs=\frac{c}{a}$. If they are easy to find, we just guess them. If they are not easy to find/guess, we simply solve instead the quadratic

$$aX^2+bX+c=0\,.$$

If this equation has two solutions, those are r and s. If the equation has one solution, then $r=s$ is the solution. if the quadratic has no solution, it means the quadratic cannot be factored.

Example 2.6.17 *Factor*

$$2X^2-5X+3\,.$$

Solution: We seek r,s so that

$$\begin{cases} r+s &= \frac{5}{2} \\ rs &= \frac{3}{2} \end{cases}\,.$$

As we cannot guess the numbers, we solve the equation

$$2X^2-5X+3=0\,.$$

$$\Delta=25-24=1\,.$$

The solutions are

$$x_{1,2}=\frac{5\pm\sqrt{1}}{4}\,,$$

which are 1 and $\frac{3}{2}$. Therefore

$$2X^2-5X+3=2(X-1)(X-\frac{3}{2}\,.$$

2.6.4 Exercises

Exercise 2.6.1 *Solve the following equations by completing the square.*

i) $x^2-8x+9=0.$

ii) $x^2+6x+10=0.$

iii) $2x^2 - 5x + 3 = 0$.

Exercise 2.6.2 *For which value of a does the following equation have exactly one solution:*

$$x^2 - 2x + a = 0\,.$$

Exercise 2.6.3 *Solve the following equations*

i) $x^2 + 4x + 3 = 0$

ii) $2x^2 + 7x - 5 = 0$

Exercise 2.6.4 *Solve the following equations*

i) $x^2 + x - 3 = 0$

ii) $3x^2 + x + 3 = 0$

Exercise 2.6.5 *Factor the following quadratics*

i) $x^2 - 7x + 12$

ii) $x^2 - 2x - 4$

Exercise 2.6.6 *Solve the following quadratic equations*

a) $3x^2 + 2x - 0$.

b) $x^2 + 3x + 7 = 0$.

Exercise 2.6.7 *Solve the following quadratic equations*

a) $x^2 + 2x - 3 = 0$.

b) $x^2 + 6x + 10 = 0$.

Exercise 2.6.8 *For which value of a does the following equation have exactly one solution:*

$$x^2 - 2x + a = 0\,.$$

2.7 Polynomials

Definition 2.7.1 *A* **polynomial** *is an expression of the form*

$$a_nX^n + a_{n-1}X^{n-1} + ... + a_1X + a_0\,,$$

where $a_n, ..., a_0$ are real numbers. These numbers are called the **coefficients** *of the polynomial.*

Example 2.7.1 *$X^2 + 2X + 1$ and $4X^4 + 2X^2 + 1$ are polynomials.*

The coefficients of $X^2 + 2x + 1$ are $1, 2, 1$, while the coefficients of $4X^4 + 2X^2 + 1$ are $4, 0, 2, 0, 1$. That is because

$$4X^4 + 2X^2 + 1 = 4X^4 + 0X^3 + 2X^2 + 0X + 1\,.$$

Note that if we don't list the zeroes, we could not say which polynomial has the coefficients $4, 2, 1$.

Definition 2.7.2 *The* **degree** *of the polynomial is the largest power of X which appears with a non-zero coefficient.*

Example 2.7.2 *Find the degrees of $X^2 + 2X + 1$, $4X^4 + 2X^2 + 1$ and 5.*

Solution: The degrees are $2, 4$ and 0. Note that $5 = 5X^0$.

We **add** two polynomials by adding the coefficients corresponding to the same power of X. We **subtract** two polynomials by subtracting the coefficients corresponding to the same power of X.

Example 2.7.3 *Find $(2X^4 + 3X^3 - X^2 + 3X + 1) + (3X^3 + X^2 - X + 1)$ and $(2X^4 + 3X^3 - X^2 + 3X + 1) - (3X^3 + X^2 - X + 1)$.*

Solution:

$$\begin{aligned}(2X^4 + 3X^3 - X^2 + 3X + 1) + (3X^3 + X^2 - X + 1)&\\ &= 2X^4 + 3X^3 - X^2 + 3X + 1 + 3X^3 + X^2 - X + 1\\ &= 2X^4 + 6X^3 + 2X + 2\end{aligned}$$

$$(2X^4 + 3X^3 - X^2 + 3X + 1) - (3X^3 + X^2 - X + 1)$$
$$= 2X^4 + 3X^3 - X^2 + 3X + 1 - 3X^3 - X^2 + X - 1$$
$$= 2X^4 - 2X^2 + 4X$$

We multiply polynomials by multiplying each term in the first polynomial by each term in the second polynomial and adding everything together.

Example 2.7.4 *Find* $(2X^3 + 3X + 1) \cdot (3X^2 + X - 1)$.

Solution:

$$(2X^3 + 3X + 1) \cdot (3X^2 + X - 1)$$
$$= 2X^3 \cdot 3X^2 + 2X^3 \cdot X + 2X^3 \cdot (-1) + 3X \cdot 3X^2 + 3X \cdot X + 3X \cdot (-1)$$
$$+ 1 \cdot 3X^2 + 1 \cdot X + 1 \cdot (-1)$$
$$= 6X^5 + 2X^4 - 2X^3 + 9X^3 + 3X^2 - 3X + 3X^2 + X - 1$$
$$= 6X^5 + 2X^4 + 7X^3 + 6X^2 - 2X - 1$$

Exercise 2.7.1 *Multiply* $(2X^3 + 3X^2 + 2) \cdot (X^2 - 2X + 1)$.

Answer: Multiplying each with each we get:

$$(2X^3 + 3X^2 + 2) \cdot (X^2 - 2X + 1) = 2X^3 \cdot X^2 + 2X^3 \cdot (-2X) + 2X^3 \cdot 1$$
$$+ 3X^2 \cdot X^2 + 3X^2 \cdot (-2X) + 3X^2 \cdot 1 + 2 \cdot X^2 + 2 \cdot (-2X) + 2 \cdot 1$$
$$= 2X^5 - 4X^4 + 2X^3 + 3X^4 - 6X^3 + 3X^2 + 2X^2 - 4X + 2$$
$$= 2X^5 - X^4 - 4X^3 + 5X^2 - 4X + 2$$

Exercise 2.7.2 *Multiply* $(2X^2 + 1)(2X^2 - 1)$.

Answer: Multiplying each with each we get:

$$(2X^2 + 1)(2X^2 - 1) = 2X^2 \cdot 2X^2 + 2X^2 \cdot (-1) + 1 \cdot 2X^2 + 1 \cdot (-1)$$
$$= 4X^4 - 2X^2 + 2X^2 - 1 = 4X^4 - 1$$

Expressions of the form sum · product appear often in algebra. For this reason it is often useful to remember the following formula:

$$\mathbf{(a+b)(a-b) = a^2 - b^2}\,.$$

Using this formula in Example 2.7.2 with $a = 2X^2$ and $b = 1$ we get

$$(2X^2+1)(2X^2-1) = (2X^2)^2 - (1)^2 = 4X^4 - 1\,.$$

Exercise 2.7.3 *Multiply* $(X+2)(X-2)$.

Answer: By the previous formula we have

$$(X+2)(X-2) = (X)^2 - (2)^2 = X^2 - 4\,.$$

Exercise 2.7.4 *Calculate* $(2X^2+1)^2$.

Answer: Multiplying each with each we get:

$$\begin{aligned}(2X^2+1)^2 &= 2X^2 \cdot 2X^2 + 2X^2 \cdot 1 + 1 \cdot 2X^2 + 1 \cdot 1\\ &= 4X^4 + 2X^2 + 2X^2 + 1 = 4X^4 + 4X^2 + 1\end{aligned}$$

Expressions of the form square of a sum or square of a difference appear often in algebra. For this reason it is often useful to remember the following formulas:

$$\mathbf{(a+b)^2 = a^2 + 2ab + b^2}\,.$$

$$\mathbf{(a-b)^2 = a^2 - 2ab + b^2}\,.$$

Using the formula for sum in Example 2.7.4 with $a = 2X^2$ and $b = 1$ we get

$$(2X^2+1)^2 = (2X^2)^2 + 2(2X^2) + (1)^2 = 4X^4 + 4X^2 + 1\,.$$

Exercise 2.7.5 *Calculate* $(X+3)^2$.

Answer: By the previous formula we have

$$(X+3)^2 = (X)^2 + 2 \cdot X \cdot 3 + 3^2 = X^2 + 6X + 9\,.$$

Exercise 2.7.6 *Calculate* $(X-2)^2$.

Answer: By the previous formula we have

$$(X-2)^2 = (X)^2 - 2X(2) + 2^2 = X^2 - 4X + 4\,.$$

Exercise 2.7.7 *Calculate* $(X^2-6)^2$

Answer: By the previous formula we have

$$(X^2-6)^2 = (X^2)^2 - 2(X^2)(6) + (6)^2 = X^4 - 12X^2 + 36\,.$$

There are also some similar formulas for cubes:

$$\mathbf{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}\,.$$

$$\mathbf{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}\,.$$

Important!: If you don't remember the formula, you can always find it by multiplying $(a \pm b)$ by itself three times.

Exercise 2.7.8 *Calculate* $(X+2)^3$.

Answer: By the previous formula we have

$$(X+2)^3 = (X)^3 + 3 \cdot X^2 \cdot 2 + 3 \cdot X \cdot 2^2 + 2^3 = X^3 + 6X^2 + 12X + 8\,.$$

Exercise 2.7.9 *Calculate* $(X-3)^3$.

Answer: By the previous formula we have

$$(X-3)^3 = (X)^3 - 3 \cdot X^2 \cdot 3 + 3 \cdot X \cdot 3^2 - 3^3 = X^3 - 9X^2 + 27X - 27\,.$$

Exercise 2.7.10 *Calculate* $(X^2-1)^3$

Answer: By the previous formula we have

$$(X^2-1)^3 = (X^2)^3 - 3 \cdot (X^2)^2 \cdot 1 + 3 \cdot X^2 \cdot 1^2 - 1^3 = X^6 - 3X^4 + 3X^2 - 1\,.$$

2.7.1 Factorization

By **factorization** we understand the following problem:

Given a polynomial P, can we find two polynomials Q, R of **smaller degree** such that

$$P(X) = Q(X) \cdot R(X)\,?$$

For example, we saw in Exercise 2.7.1 that

$$(2X^3 + 3X^2 + 2) \cdot (X^2 - 2X + 1) = 2X^5 - X^4 - 4X^3 + 5X^2 - 4X + 2$$

This is telling us that the polynomial $2X^5 - X^4 - 4X^3 + 5X^2 - 4X + 2$ can be factored as $(2X^3 + 3X^2 + 2) \cdot (X^2 - 2X + 1)$.

We might also ask if $(2X^3 + 3X^2 + 2)$ or $(X^2 - 2X + 1)$ can be factored further, this is what we understand by complete factorization.

Finding the factorization of a polynomial in general is a very hard question, but in some situations we can find the factorization.

2.7.2 Review: Factorization of quadratics

Let us recall the following fact from the section about quadratic Equations:

Theorem 2.7.1 *Consider the polynomial*

$$aX^2 + bX + c\,.$$

Let $\Delta = b^2 - 4ac$.

- *i) If* $\Delta < 0$ *then* $aX^2 + bX + c$ *cannot be factored.*
- *ii) If* $\Delta = 0$ *then* $aX^2 + bX + c$ *be factored as*

  $$aX^2 + bX + c = a(X - r)^2\,.$$

  where $r = -\frac{b}{2a}$ *is the solution to* $aX^2 + bX + c = 0$.
- *iii) If* $\Delta > 0$ *then* $aX^2 + bX + c$ *be factored as*

  $$aX^2 + bX + c = a(X - r_1)(X - r_2)\,.$$

  where $r_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$ *are the solutions to* $aX^2 + bX + c = 0$.

Exercise 2.7.11 *Factor* $3X^2 - 2X + 1$.

Answer:

$$\Delta = 4 - 12 < 0\,.$$

This cannot be factored.

Exercise 2.7.12 *Factor* $X^2 - 2X + 1$.

Answer:

$$\Delta = 0\,.$$

The solution to $X^2 - 2X + 1 = 0$ is $X = 1$. Therefore

$$X^2 - 2X + 1 = (X - 1)^2\,.$$

Exercise 2.7.13 *Factor* $X^2 - 5X + 6$.

Answer:

$$\Delta = 25 - 24 > 0\,.$$

The solutions to $X^2 - 5X + 6 = 0$ are $r_{1,2} = \frac{5 \pm \sqrt{1}}{2}$, which are 2 and 3 . Therefore

$$X^2 - 5X + 6 = (X - 2)(X - 3)\,.$$

2.7.3 Factorization by common factor

If the Polynomial doesn't contain a term without X, we can factor it by making the smallest power of X a common factor.

Exercise 2.7.14 *Factor* $3X^2 - 2X$.

Answer:

$$3X^2 - 2X = X(3X - 2)\,.$$

Exercise 2.7.15 *Factor* $X^3 - 2X^2$.

Answer:

$$X^3 - 2X^2 = X^2(X - 3)\,.$$

Exercise 2.7.16 *Factor* $X^4 - 5X^3 + 4X^2$.

Answer:

$$X^4 - 5X^3 + 4X^2 = X^2(X^2 - 5X + 4)\,.$$

Now we can factor further the quadratic $X^2 - 5X + 4$:

$$\Delta = 25 - 16 = 9\,.$$

The roots are $r_{1,2} = \frac{5 \pm \sqrt{9}}{2}$ which is 1 and 4. Therefore $X^2 - 5X + 4 = (X-1)(X-4)$.

This gives

$$X^4 - 5X^3 + 4X^2 = X^2(X-1)(X-4)\,.$$

2.7.4 Factorization by Formulas

The following formulas can also be used for factorization:

$$\mathbf{a^2 - b^2 = (a+b)(a-b)}\,.$$

$$\mathbf{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}\,.$$

$$\mathbf{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}\,.$$

Exercise 2.7.17 *Factor* $X^2 - 4$.

Answer:

$$X^2 - 4 = X^2 - 2^2 = (X-2)(X+2)\,.$$

Exercise 2.7.18 *Factor* $X^2 - 7$.

Answer:

$$X^2 - 7 = X^2 - (\sqrt{7})^2 = (X - \sqrt{7})(X + \sqrt{7})\,.$$

Exercise 2.7.19 *Factor* $X^3 - 8$.

Answer:

$$X^3 - 8 = X^3 - (2)^3 = (X - 2)(X^2 + 2X + 4)\,.$$

The quadratic $X^2 + 2X + 4$ has $\Delta = 4 - 16 < 0$, therefore we cannot factor further.

Exercise 2.7.20 *Factor* $X^3 + 1$.

Answer:

$$X^3 + 1 = X^3 + (1)^3 = (X + 1)(X^2 - X + 1)\,.$$

The quadratic $X^2 - X + 1$ has $\Delta = 1 - 4 < 0$, therefore we cannot factor further.

Exercise 2.7.21 *Factor* $X^4 - 16$.

Answer:

$$X^4 - 16 = (X^2)^2 - 4^2 = (X^2 - 4)(X^2 + 4) = (X^2 - 2^2)(X^2 + 4) = (X - 2)(X + 2)(X^2 + 4)\,.$$

2.7.5 Long Division

Exactly as for numbers, we can do long division with Polynomials.

Theorem 2.7.2 *Given two polynomials $P(X)$ and $D(X)$, there exists unique polynomials $Q(X)$ and $R(X)$ such that*

$$P(X) = Q(X) \cdot D(X) + R(X)\,,$$

and either $R(X) = 0$ or $\deg(R) < \deg(D)$.

When we perform long division of polynomials we proceed as follows:

We start by dividing the term in $P(X)$ with the highest degree by the term in $D(X)$ with the highest degree. Their ratio yields a new term for $Q(X)$.

Next we multiply this term by $D(X)$ and write it under $P(X)$. We subtract this from $P(X)$.

As long as the difference we get has degree greater or equal than $D(X)$, we repeat the above process with the difference instead of $P(X)$.

Exercise 2.7.22 *Divide* $X^4 + 2X^3 + 3X^2 + 2X - 1$ *by* $X^2 + X + 1$.

Answer:

The highest term in $X^4 + 2X^3 + 3X^2 + 2X - 1$ is X^4. The highest term in $X^2 + X + 1$ is X^2. Dividing X^4 by X^2 we get $\frac{X^4}{X^2} = X^2$. This is the first term in $Q(X)$.

So far we have

$$\begin{array}{r|l} & X^2 \\ \hline X^2 + X + 1 & X^4 + 2X^3 + 3X^2 + 2X - 1 \end{array}$$

Next we subtract $X^2(X^2 + X + 1) = X^4 + X^3 + X^2$.

$$\begin{array}{r|l} & X^2 \\ \hline X^2 + X + 1 & X^4 + 2X^3 + 3X^2 + 2X - 1 \\ & \underline{X^4 + X^3 + X^2} \end{array}$$

We now subtract

$$(X^4 + 2X^3 + 3X^2 + 2X - 1) - (X^4 + X^3 + X^2) .$$

$$\begin{array}{r|l} & X^2 \\ \hline X^2 + X + 1 & X^4 + 2X^3 + 3X^2 + 2X - 1 \\ & \underline{X^4 + \ \ X^3 + \ \ X^2} \\ & \qquad\quad X^3 + 2X^2 + 2X - 1 \end{array}$$

Dividing X^3 by X^2 we get $\frac{X^3}{X^2} = X$. This is the second term in $Q(X)$.

So far we have

$$\begin{array}{r|l} & X^2 + X \\ \hline X^2 + X + 1 & X^4 + 2X^3 + 3X^2 + 2X - 1 \\ & \underline{X^4 + \ \ X^3 + \ \ X^2} \\ & \qquad\quad X^3 + 2X^2 + 2X - 1 \end{array}$$

Next we subtract $X(X^2 + X + 1) = X^3 + X^2 + X$.

$$
\begin{array}{r}
X^2 + X \\
X^2 + X + 1 \overline{) X^4 + 2X^3 + 3X^2 + 2X - 1} \\
\underline{X^4 + X^3 + X^2} \\
X^3 + 2X^2 + 2X - 1 \\
\underline{X^3 + X^2 + X}
\end{array}
$$

We now subtract

$$(X^3 + 2X^2 + 2X - 1) - (X^3 + X^2 + X) .$$

$$
\begin{array}{r}
X^2 + X \\
{\color{red}X^2} + X + 1 \overline{) X^4 + 2X^3 + 3X^2 + 2X - 1} \\
\underline{X^4 + X^3 + X^2} \\
X^3 + 2X^2 + 2X - 1 \\
\underline{X^3 + X^2 + X} \\
{\color{red}X^2} + X - 1
\end{array}
$$

Dividing X^2 by X^2 we get $\frac{X^2}{X^2} = {\color{red}1}$. This is the second term in $Q(X)$. So far we have

$$
\begin{array}{r}
X^2 + X + {\color{red}1} \\
{\color{red}X^2} + X + 1 \overline{) X^4 + 2X^3 + 3X^2 + 2X - 1} \\
\underline{X^4 + X^3 + X^2} \\
X^3 + 2X^2 + 2X - 1 \\
\underline{X^3 + X^2 + X} \\
{\color{red}X^2} + X - 1
\end{array}
$$

Next we subtract $1(X^2 + X + 1) = X^2 + X + 1$.

$$
\begin{array}{r}
X^2 + X + 1 \\
X^2 + X + 1 \overline{) X^4 + 2X^3 + 3X^2 + 2X - 1} \\
\underline{X^4 + X^3 + X^2} \\
X^3 + 2X^2 + 2X - 1 \\
\underline{X^3 + X^2 + X} \\
X^2 + X - 1 \\
\underline{X^2 + X + 1}
\end{array}
$$

We now subtract

$$(X^2+X-1)-(X^2+X+1)\,.$$

$$\begin{array}{r l}
 & \phantom{X^4+2X^3+{}}X^2+X+1 \\ \cline{2-2}
X^2+X+1 \big) & X^4+2X^3+3X^2+2X-1 \\
 & \underline{X^4+\ X^3+\ X^2} \\
 & \phantom{X^4+{}}X^3+2X^2+2X-1 \\
 & \phantom{X^4+{}}\underline{X^3+\ X^2+\ X} \\
 & \phantom{X^4+X^3+{}}X^2+X-1 \\
 & \phantom{X^4+X^3+{}}\underline{X^2+X+1} \\
 & -2
\end{array}$$

Answer: The quotient is $Q(X)=X^2+X+1$ and the remainder is $R(X)=-2$.

Note that we can check our answer:

$$X^4+2X^3+3X^2+2X-1=(X^2+X+1)(X^2+X+1)-2\checkmark\,.$$

Exercise 2.7.23 *Divide* $X^4+3X^3+X^2-2X+1$ *by* X^2-3X+1.

Answer:

$$\begin{array}{r l}
 & \phantom{X^4+3X^3+{}}X^2+6X+18 \\ \cline{2-2}
X^2-3X+1 \big) & X^4+3X^3+X^2-\ 2X+\ 1 \\
 & \underline{X^4-3X^3+X^2} \\
 & \phantom{X^4+{}}6X^3-\qquad\ \ 2X+\ 1 \\
 & \phantom{X^4+{}}\underline{6X^3-18X^2+6X} \\
 & \phantom{X^4+6X^3+{}}18X^2-8X+\ 1 \\
 & \phantom{X^4+6X^3+{}}\underline{18X^2-54X+18} \\
 & \phantom{X^4+6X^3+18X^2-{}}46X-17
\end{array}$$

Answer: The quotient is $Q(X)=X^2+6X+18$ and the remainder is $R(X)=46X-17$.

Note that we can check our answer:

$$X^4+3X^3+X^2-2X+1=(X^2-3X+1)(X^2+6X+18)+46X-17\checkmark\,.$$

Exercise 2.7.24 *Divide* X^3+3X^2+3X+1 *by* $X+1$.

Answer:

$$
\begin{array}{r}
X^2 + 2X + 1 \\
X+1 \overline{) X^3 + 3X^2 + 3X + 1} \\
\underline{X^3 + X^2} \qquad\qquad\quad \\
2X^2 + 3X + \ \ 1 \\
\underline{2X^2 + 2X} \quad\ \ \\
X + 1 \\
\underline{X + 1} \\
0 \qquad\qquad
\end{array}
$$

Answer: The quotient is $Q(X) = X^2 + 2X + 1$ and the remainder is $R(X) = 0$.

Note that we can check our answer:

$$X^3 + 3X^2 + 3X^2 + 1 = (X + 1)(X^2 + 2X + 1)\checkmark\ .$$

Important!: Whenever when we get a remainder of 0, we actually get a factorization of $P(X)$.

In the next exercise we will see that if the highest degree term of $D(X)$ has a coefficient not equal to 1, even if all the coefficients of $P(X)$ are integers, we often get fractions in $Q(X)$ and $R(X)$.

Exercise 2.7.25 *Divide* $X^3 + X^2 + 3X + 1$ *by* $2X + 1$.

Answer:

$$
\begin{array}{r}
\frac{1}{2}X^2 + \frac{1}{4}X + \frac{11}{8} \\
2X+1 \overline{) X^3 + X^2 + 3X + 1} \\
\underline{X^3 + \frac{1}{2}X^2} \qquad\qquad\ \ \\
\frac{1}{2}X^2 + 3X + \ \ 1 \\
\underline{\frac{1}{2}X^2 + \frac{1}{4}X} \quad\ \ \\
\frac{11}{4}X + 1 \\
\underline{\frac{11}{4}X + \frac{11}{8}} \\
-\frac{3}{8} \qquad\qquad
\end{array}
$$

Answer: The quotient is $Q(X) = \frac{1}{2}X^2 + \frac{1}{4}X + \frac{11}{8}$ and the remainder is $R(X) = -\frac{3}{8}$.

2.7.6 Rational Root Test

Definition 2.7.3 *A real number a is called a* **root** *for the polynomial $P(X)$ if $P(a) = 0$.*

Exercise 2.7.26 *Are $0, 1, 2, 3$ roots of $P(X) = X^3 - 2X^2 - X - 6$?*

Answer:

$$P(0) = 0 - 0 - 0 - 6 \neq 0\, X\,.$$

$$P(1) = 1 - 2 - 1 - 6 \neq 0\, X\,.$$

$$P(2) = 8 - 8 - 2 - 6 \neq 0\, X\,.$$

$$P(3) = 27 - 18 - 3 - 600 \checkmark\,.$$

$0, 1, 2$ are not roots of $P(X)$. 3 is a root of $P(X)$.

Theorem 2.7.3 *If P is not the zero polynomial, then $P(X)$ has at most $\deg(P)$ roots.*

Important!: To find the roots of a polynomial of degree 1 or 2, we simply solve the linear or quadratic equation $P(X) = 0$.

The question is how can we find the roots of higher degree polynomials? If all the coefficients are integers then we can find the rational roots.

Theorem 2.7.4 *(***Rational Root Test***) Let $P(X) = a_n X^n + ... + a_1 X + a_0$ be a polynomial with all the coefficients integers. If $r = \frac{m}{l} \neq 0$ is a rational root, then $m|a_0$ and $l|a_n$.*

In particular, if $a_n = 1$ then all the rational roots of $P(X)$ are integers and divisors of a_0.

Important!: Remember that r can be negative.

Exercise 2.7.27 *Find all rational roots of $P(X) = X^3 - 3X^2 + 4X - 2$.*

Answer: By the Rational root test the potential roots are ± 1 and ± 2. We now test them

$$P(1) = 0\checkmark\,.$$

$$P(-1) \neq 0X\,.$$

$$P(2) \neq 0X\,.$$

$$P(-2) \neq 0X\,.$$

The only rational root is 1.

Exercise 2.7.28 *Find all rational roots of* $P(X) = 2X^3 - X^2 - 2X + 1$.

Answer: By the Rational root test the potential roots are ± 1 and $\pm \frac{1}{2}$. We now test them

$$P(1) = 0\checkmark\,.$$

$$P(-1) = 0\checkmark\,.$$

$$P(\frac{1}{2}) = 0\checkmark\,.$$

$$P(-\frac{1}{2}) \neq 0X\,.$$

The rational roots are $1, -1, -\frac{1}{2}$.

2.7.7 Rational Roots and Factorisation

Note that if we have a root of $P(X)$ we can factor it:

Theorem 2.7.5 *a is a root of $P(X)$ if and only if $(X - a)$ is a factor of $P(X)$.*

Thus, if we can find a rational root of $P(X)$, we can then do long division of $P(X)$ by $(X - a)$. The remainder has to be zero, and if possible we try to factor the quotient.

Exercise 2.7.29 *Factor* $X^3 - 3X + 2$.

By the Rational root test the potential roots are ± 1 and ± 2. By checking, we find that 1 is a root.

Dividing $X^3 - 3X^2 + 2$ by $X - 1$ we get a quotient of $X^2 - 3X + 2$ and a remainder of $R = 0$. Therefore

$$(X^3 - 3X^2 + 2) = (X - 1)(X^2 - 3X + 2)\,.$$

Now, the roots of the quadratic $X^2 - 3X + 2 = 0$ are 1 and 2. Therefore $X^2 - 3X + 2 = (X - 1)(X - 2)$. This Yields:

$$(X^3 - 3X^2 + 2) = (X - 1)(X - 1)(X - 2)\,.$$

Exercise 2.7.30 *Factor* $X^3 - X^2 + X - 1$.

By the Rational root test the potential roots are ± 1. By checking, we find that 1 is a root.

Dividing $X^3 - X^2 + X - 1$ by $X - 1$ we get a quotient of $X^2 + 1$ and a remainder of $R = 0$. Therefore

$$(X^3 - X^2 + X - 1) = (X - 1)(X^2 + 1)\,.$$

Now, the quadratic $X^2 + 1 = 0$ doesn't have solutions, which means this cannot be factored further.

2.7.8 Exercises

Exercise 2.7.31 *Calculate*

i) $(X^2 + 2X + 1) + (X^3 + 2X - 1)$.

ii) $(X^2 + 2X + 1) \cdot (X^3 + 2X - 1)$

Exercise 2.7.32 *Calculate*

i) $(2X + 1)^2$

ii) $(X^2 - 1)^2$

iii) $(X + 2)^3$

Exercise 2.7.33 *Factor*

i) $X^2 - 7X + 12$

ii) $3X^3 + 2X^2$

iii) $X^4 - 81$

Exercise 2.7.34 *Find the quotient and remainder for the following long divisions:*

i) $X^4 - 3X^3 + 2X^2 - X + 1$ *divided by* $X^2 + 2X + 1$*.*

ii) $2X^3 + 3X^2 - X + 1$ *divided by* $2X + 1$*.*

Exercise 2.7.35 *Find all rational roots of*

$$P(X) = 3X^3 + 2X^2 + 5X - 2\,.$$

Exercise 2.7.36 *Factor*

$$X^3 - X^3 - 4x + 4\,.$$

Exercise 2.7.37 *Find the quotient and remainder when* $X^4 - 3X^3 - 2X^2 + 2$ *is divided by* $X^2 + 2X + 3$*.*

Exercise 2.7.38 *Find the quotient and remainder when* $2X^4 - 2X^3 + 3X^2 - 2$ *is divided by* $X^2 + 2X + 2$*.*

Exercise 2.7.39 *Find the quotient and remainder when* $X^4 + 3X^2 + 2X + 1$ *is divided by* $X^2 + X - 1$*.*

Exercise 2.7.40 *Find the rational roots of* $X^3 - 2X^2 + X - 2$*.*

2.8 Rationalization

By rationalization we understand the process of eliminating a square root from a denominator.

Fractions with denominator of the form $\sqrt{a}$.

If a is an integer and a fraction has the denominator $\sqrt{a}$, to rationalize the fraction we multiply both the denominator and numerator by $\sqrt{a}$.

Example 2.8.1 *Rationalize* $\frac{1}{\sqrt{7}}$.

Solution:

$$\frac{1}{\sqrt{7}} = \frac{1}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{7}}{7}\,.$$

Exercise 2.8.1 *Rationalize* $\frac{3}{\sqrt{3}}$.

Solution:

$$\frac{3}{\sqrt{3}} = \frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{7}}{3} = \sqrt{3}\,.$$

Exercise 2.8.2 *Rationalize* $\frac{1}{\sqrt{2}}$.

Solution

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}\,.$$

Exercise 2.8.3 *Rationalize* $\frac{3+\sqrt{5}}{\sqrt{5}}$.

Solution:

$$\frac{3+\sqrt{5}}{\sqrt{5}} = \frac{3+\sqrt{5}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{(3+\sqrt{5})\sqrt{5}}{5} = \frac{3\sqrt{5}+5}{5}\,.$$

Fractions with denominator of the form $a \pm \sqrt{b}$, $\sqrt{a} \pm b$, $\sqrt{a} \pm \sqrt{b}$.

Let us recall that

$$(x+y)(x-y) = x^2 - y^2\,.$$

This means that

$$(a+\sqrt{b})(a-\sqrt{b}) = a^2 - b\,.$$
$$(\sqrt{a}+b)(\sqrt{a}-b) = a - b^2\,.$$
$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a - b\,.$$

The pairs $\left(a+\sqrt{b}, a-\sqrt{b}\right)$, $\left(\sqrt{a}+b, \sqrt{a}-b\right)$ and $\left(\sqrt{a}+\sqrt{b}, \sqrt{a}-\sqrt{b}\right)$ are called **conjugated pairs**.

If a fraction has the denominator of the form $a\pm\sqrt{b}$, $\sqrt{a}\pm b$, $\sqrt{a}\pm\sqrt{b}$, with a, b integers, to rationalize we multiply both the denominator and numerator by the conjugate of the denominator.

Example 2.8.2 *Rationalize* $\frac{1}{\sqrt{7}}$.

Solution:

$$\frac{1}{\sqrt{5}-2} = \frac{1}{\sqrt{5}-2}\cdot\frac{\sqrt{5}+2}{\sqrt{5}2} = \frac{\sqrt{5}+2}{(\sqrt{5})^2-2^2} = \frac{\sqrt{5}+2}{5-4} = \sqrt{5}+2\,.$$

Exercise 2.8.4 *Rationalize* $\frac{2}{\sqrt{7}+3}$.

Solution:

$$\frac{2}{\sqrt{7}+3} = \frac{2}{\sqrt{7}+3}\cdot\frac{\sqrt{7}-3}{\sqrt{7}-3} = \frac{2(\sqrt{7}-3)}{7-9} = \frac{2(\sqrt{7}-3)}{-2} = -(\sqrt{7}-3) = 3-\sqrt{7}\,.$$

Exercise 2.8.5 *Rationalize* $\frac{\sqrt{2}}{\sqrt{5}-\sqrt{2}}$.

Solution:

$$\frac{\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{5}-\sqrt{2}}\cdot\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}} = \frac{\sqrt{10}-2}{3}\,.$$

Exercise 2.8.6 *Rationalize* $\frac{3}{\sqrt{x}-1}$.

Solution:

$$\frac{3}{\sqrt{x}-1} = \frac{3}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1} = \frac{3(\sqrt{x}+1)}{x-1}\,.$$

Exercise 2.8.7 *Rationalize* $\frac{2}{\sqrt{x+1}+\sqrt{x-1}}$.

Solution:

$$\begin{aligned}\frac{2}{\sqrt{x+1}+\sqrt{x-1}} &= \frac{2}{\sqrt{x+1}+\sqrt{x-1}} \cdot \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} \\ &= \frac{2(\sqrt{x+1}-\sqrt{x-1})}{(x+1)-(x-1)} = \frac{2(\sqrt{x+1}-\sqrt{x-1})}{2} \\ &= \sqrt{x+1}-\sqrt{x-1}\,.\end{aligned}$$

2.8.1 Exercises

Exercise 2.8.8 *Rationalize*

i) $\frac{2}{\sqrt{7}}$

ii) $\frac{1}{\sqrt{3}-1}$

iii) $\frac{2}{\sqrt{7}-\sqrt{3}}$

Exercise 2.8.9 *Rationalize* $\frac{3}{\sqrt{3}-1}$.

Exercise 2.8.10 *Rationalize* $\frac{2}{\sqrt{5}-\sqrt{2}}$.

Exercise 2.8.11 *Rationalize* $\frac{1}{\sqrt{3}-2}$.

2.9 Sketching Curves

By a curve we understand in general an equation of the form $y = f(x)$ or more generally of the form $F(x, y) = 0$.

By drawing the curve we understand plotting all pair of points (x, y) which satisfy the equation. Note that usually there are infinitely many of them.

2.9.1 Lines: $y = mx + b$

If m, b are real numbers, the equation $y = mx + b$ defines a line.

m is the **slope**, and it tells us how fast the line is raising (if $m > 0$) or decreasing (if $m < 0$).

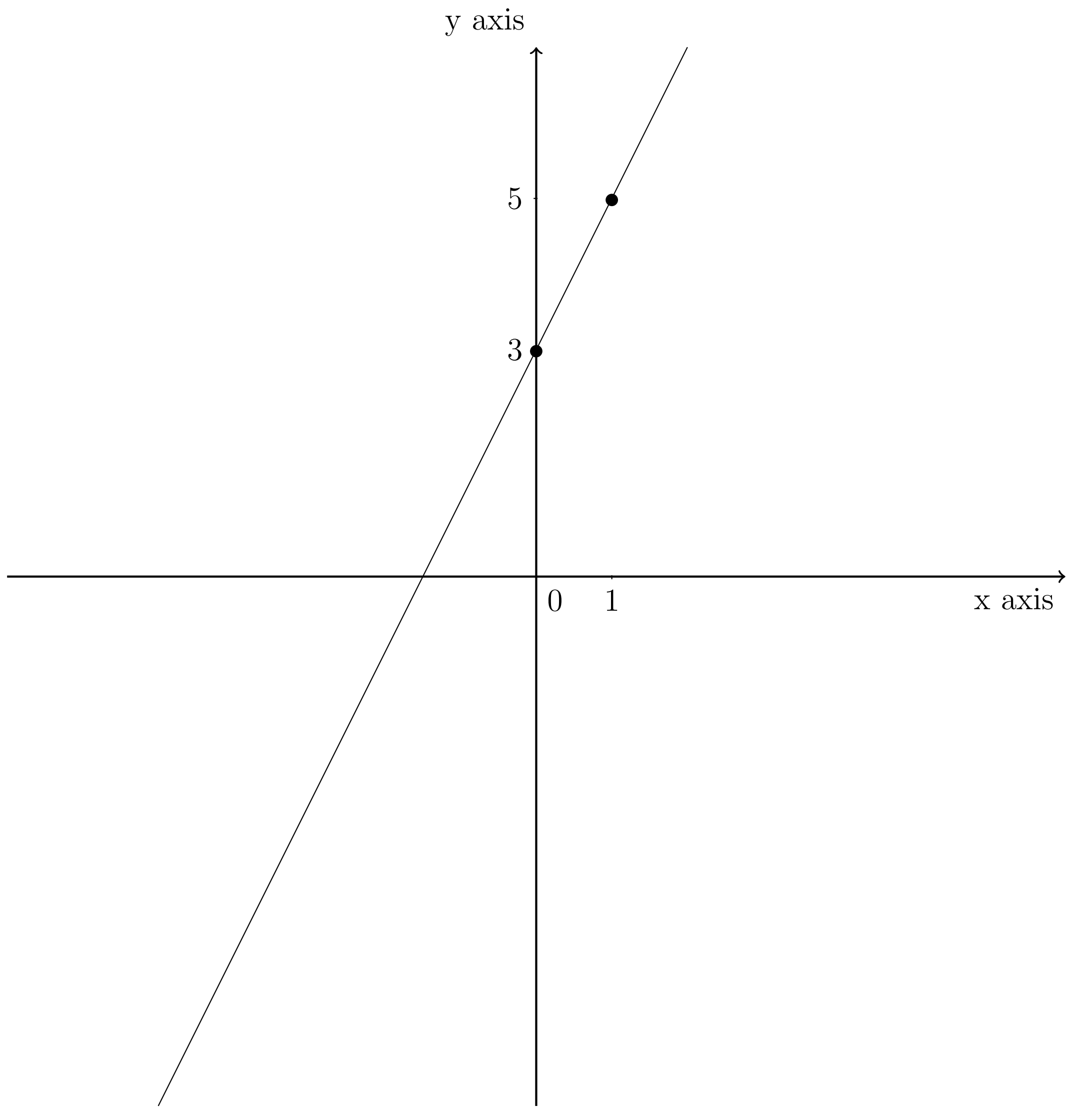
If $m = 0$ the line is horizontal.

b is the y-intercept, it tells us where the line intersects the y-axis.

To draw any line, we can plot two points and draw the line through those points.

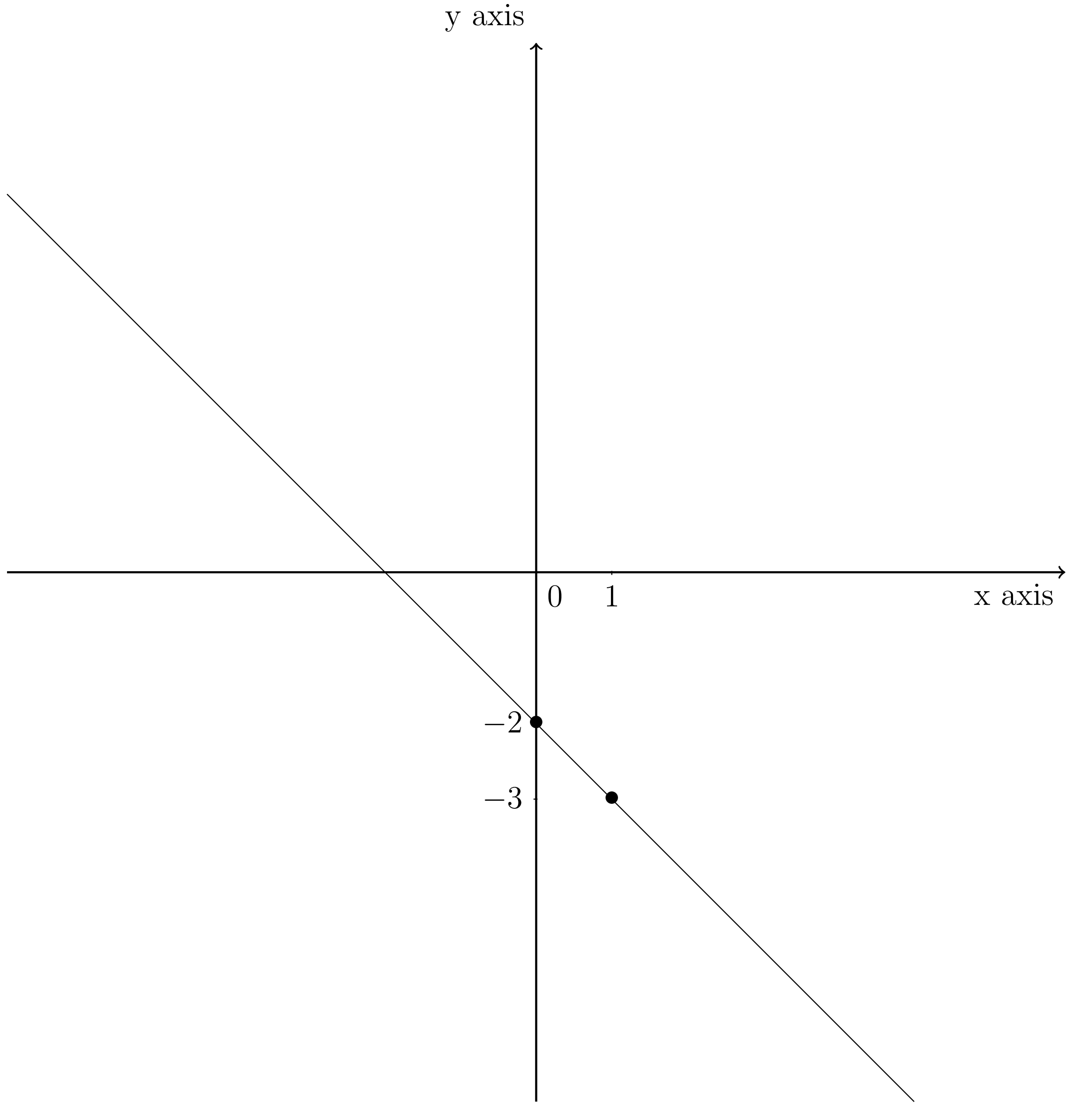
Exercise 2.9.1 *Sketch* $y = 2x + 3$.

When $x = 0, y = 3$. Thus $(0, 3)$ is a point on the line.
When $x = 1, y = 5$. Thus $(1, 5)$ is a point on the line.



Exercise 2.9.2 *Sketch* $y = -x - 2$.

When $x = 0, y = -2$. Thus $(0, -2)$ is a point on the line.
When $x = 1, y = -3$. Thus $(1, -3)$ is a point on the line.



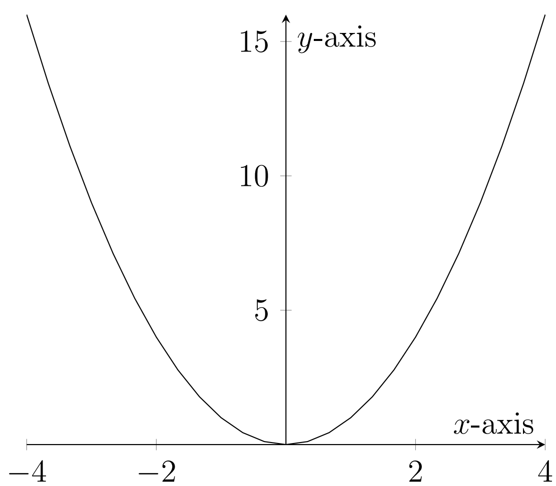
Note: When we solve a linear equation

$$ax + b = 0\,,$$

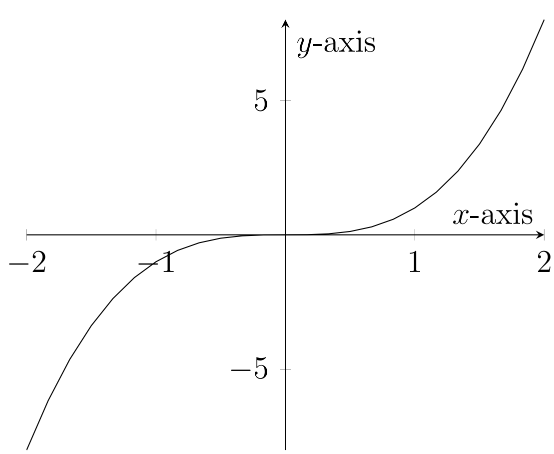
we find the intersection between the line $y = ax + b$ and the y-axis.

2.9.2 $y = x^n\,;\, n \geq 2$

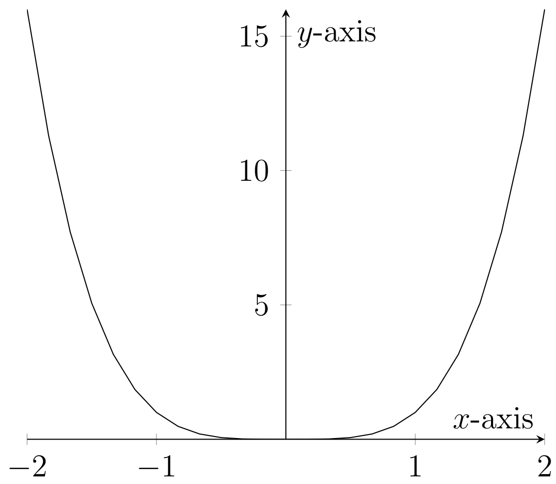
The graph of $y = x^2$ is a parabola:



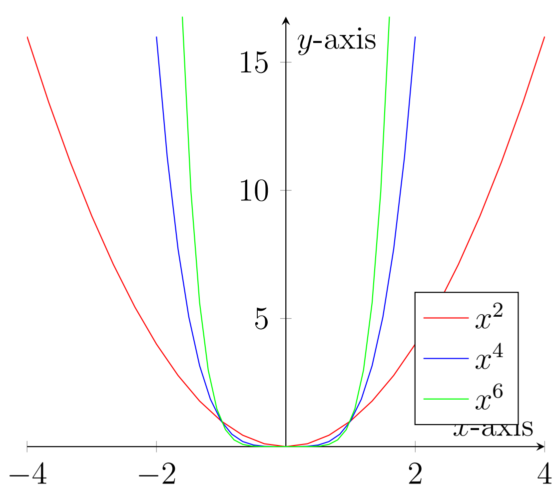
The graph of $y = x^3$ looks like this:



The graph of $y = x^4$ looks like this:

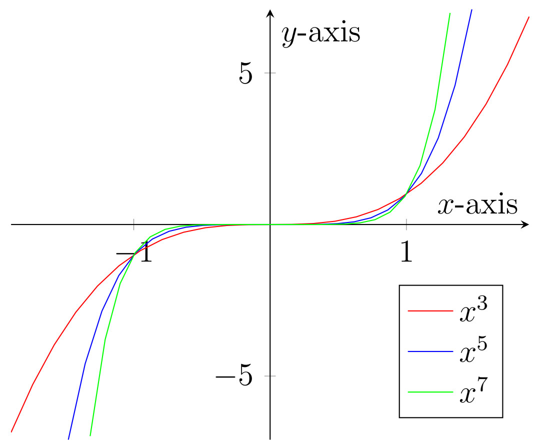


To compare, here are the graphs of x^2, x^4, x^4 drawn on the same graph:



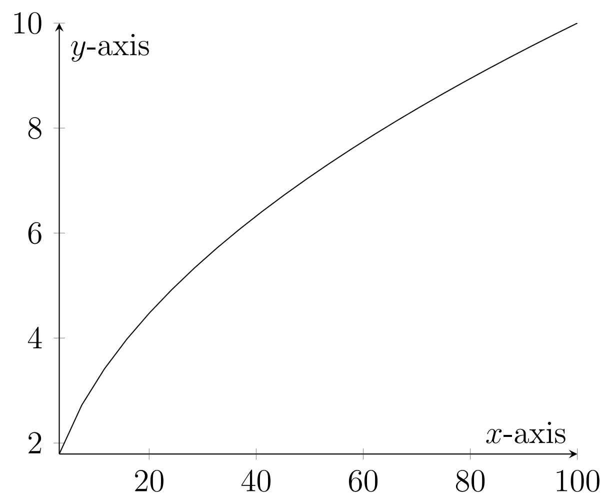
Note: Only the graph of $y = x^2$ is a parabola.

Also, below are the graphs of x^3, x^5, x^7 drawn on the same graph:



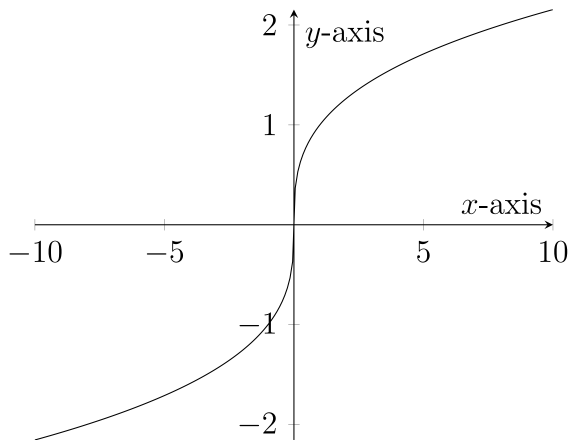
2.9.3 $y = \sqrt{x}, y = \sqrt[3]{x}$

The graph of $y = \sqrt{x}$ is half parabola which opens along the x-axis:



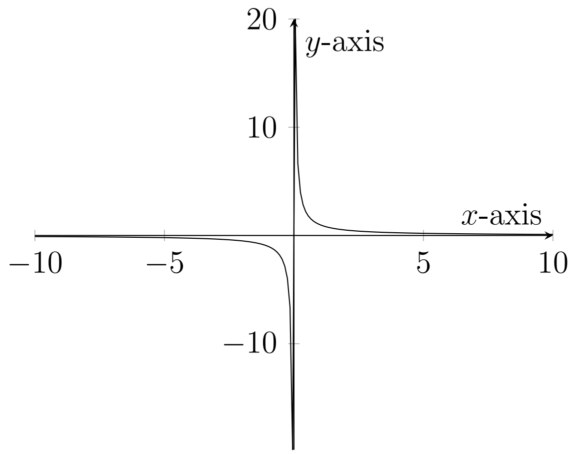
Note: The reason why this is half parabola is easy to see: the equation $y = \sqrt{x}$ is equivalent to $x = y^2$; $y \geq 0$. If we think about it as x as function in y, this is the positive half of the parabola given by $f(y) = y^2$.

The graph of $y = \sqrt[3]{x}$ is given below:



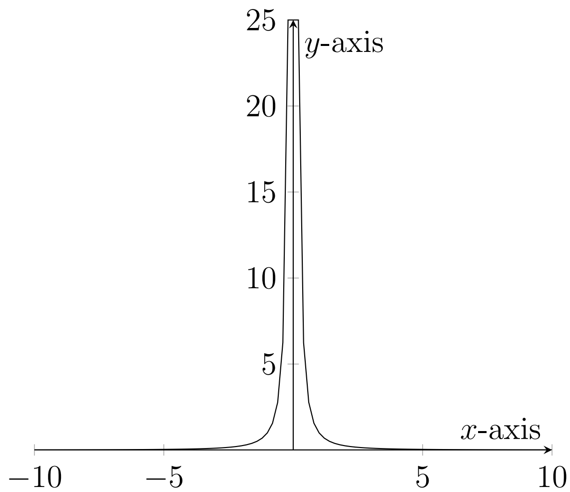
2.9.4 $y = \frac{1}{x}, y = \frac{1}{x^2}$

The graph of $y = \frac{1}{x}$ is given below:



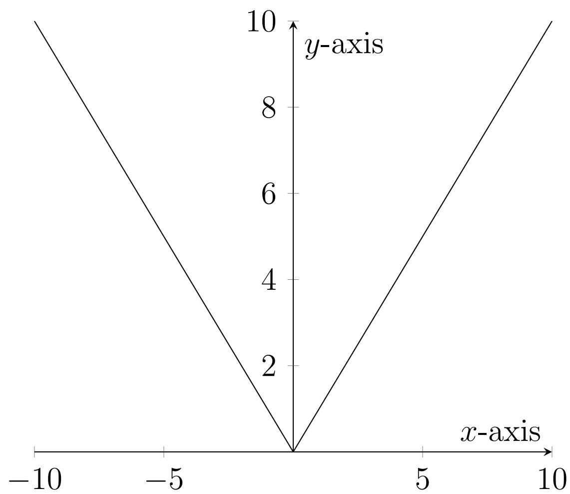
Note: The shape of this curve is called a **hyperbola**.

The graph of $y = \frac{1}{x^2}$ is given below:



2.9.5 $y = |x|$

The graph of $y = |x|$ is given below:



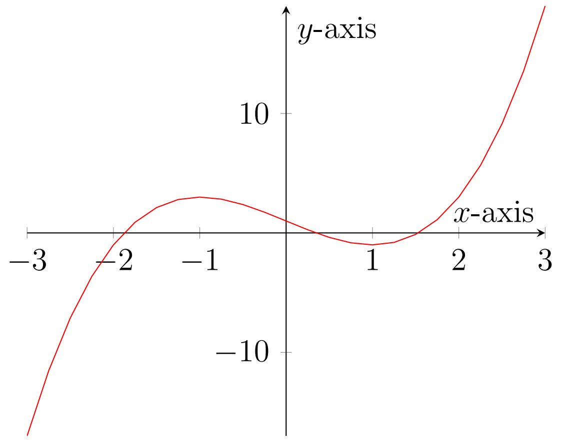
2.9.6 Operations on graphs

The graph of $f(x \pm a)$.

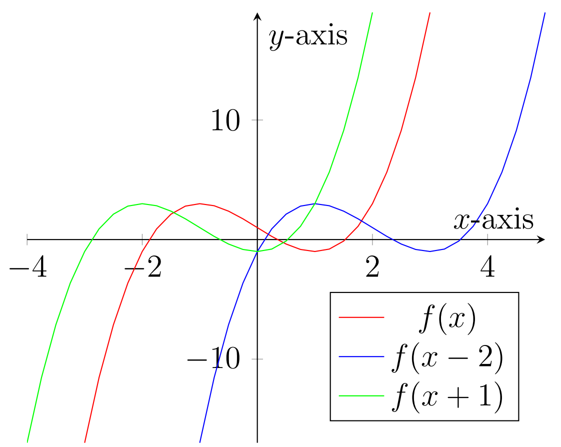
If $a > 0$ the graph of $f(x - a)$ is obtained from the graph of $f(x)$ by moving a spots to the right.

If $a > 0$ the graph of $f(x+a)$ is obtained from the graph of $f(x)$ by moving a spots to the left.

For example, if the graph of $f(x)$ is given below

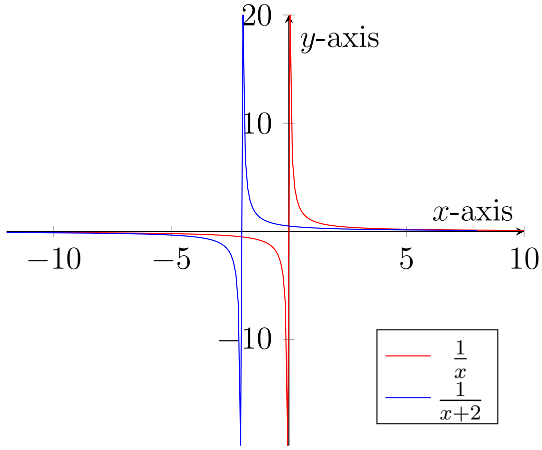


then, the graphs of $f(x-2)$ and $f(x+1)$ are



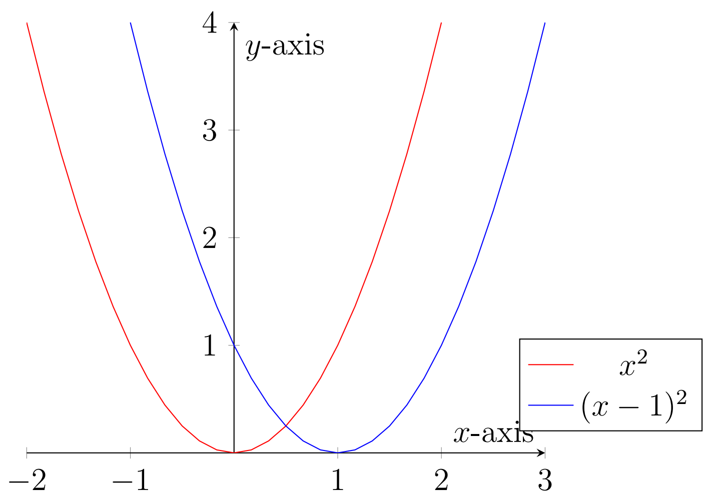
Exercise 2.9.3 *Sketch the graph of* $\frac{1}{x+2}$.

If $f(x) = \frac{1}{x}$ we have $f(x+2) = \frac{1}{x+2}$. Thus, we start with the graph of $f(x) = \frac{1}{x}$ and move it two units to the left:



Exercise 2.9.4 *Sketch the graph of* $(x-1)^2$.

If $f(x) = x^2$ we have $f(x-1) = (x-1)^2$. Thus, we start with the graph of $f(x) = x^2$ and move it one unit to the right:

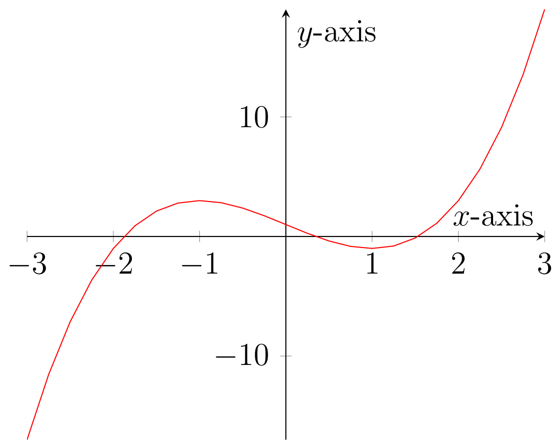


The graph of $f(x) \pm a$.

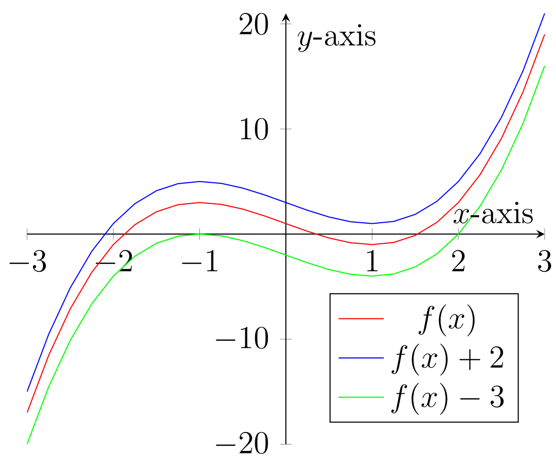
If $a > 0$ the graph of $f(x) + a$ is obtained from the graph of $f(x)$ by moving a spots up.

If $a > 0$ the graph of $f(x - a)$ is obtained from the graph of $f(x)$ by moving a spots down.

For example, if the graph of $f(x)$ is given below

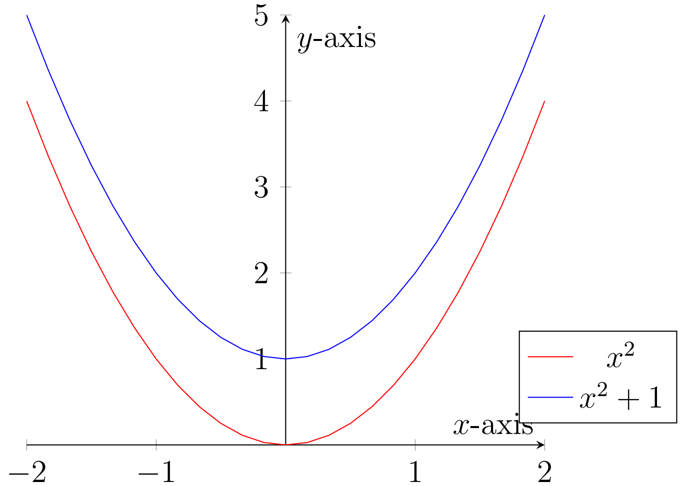


then, the graphs of $f(x)+2$ and $f(x)-3$ are



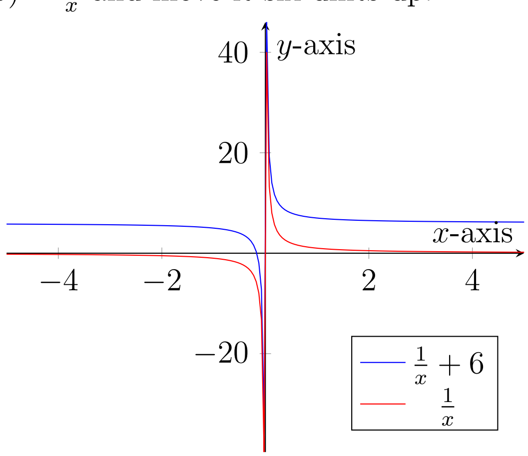
Exercise 2.9.5 *Sketch the graph of* x^2+1.

If $f(x)=x^2$ we have $f(x)+1=x^2+1$. Thus, we start with the graph of $f(x)=x^2$ and move it one unit up:



Exercise 2.9.6 *Sketch the graph of* $\frac{1}{x} + 6$.

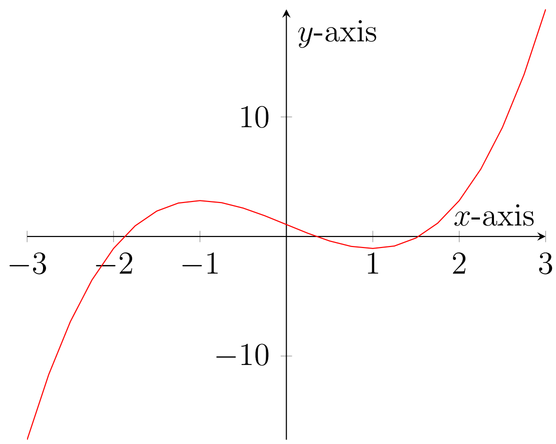
If $f(x) = \frac{1}{x}$ we have $f(x) + 2 = \frac{1}{x} + 6$. Thus, we start with the graph of $f(x) = \frac{1}{x}$ and move it six units up:



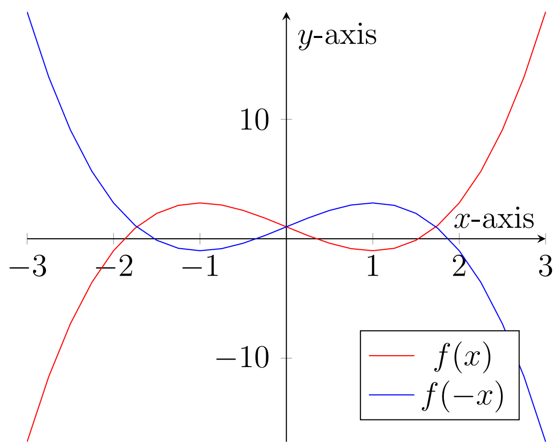
The graph of $f(-x)$.

The graph of $f(-x)$ is obtained from the graph of $f(x)$ reflection in the y-axis.

For example, if the graph of $f(x)$ is given below

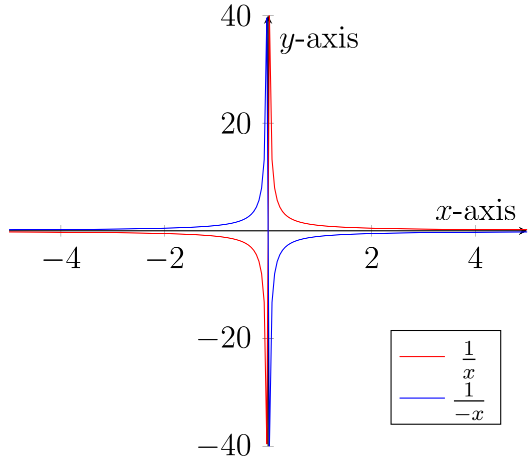


then, the graphs of $f(-x)$ is



Exercise 2.9.7 *Sketch the graph of* $\frac{1}{-x}$.

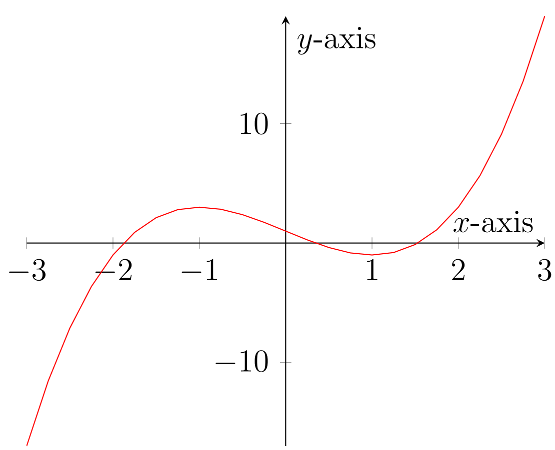
If $f(x) = \frac{1}{x}$ we have $f(-x) = \frac{1}{-x}$. Thus, we start with the graph of $f(x) = \frac{1}{x}$ and we reflect it in the y-axis:



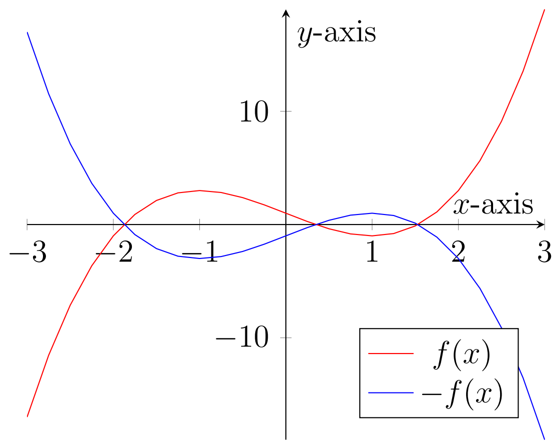
The graph of $-f(x)$.

The graph of $-f(x)$ is obtained from the graph of $f(x)$ reflection in the x-axis.

For example, if the graph of $f(x)$ is given below

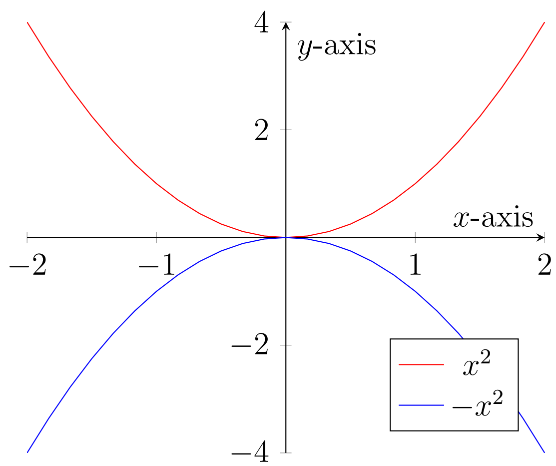


then, the graphs of $-f(x)$ is



Exercise 2.9.8 *Sketch the graph of* $-x^2$.

If $f(x) = x^2$ we have $-f(x) = -x^2$. Thus, we start with the graph of $f(x) = x^2$ and we reflect it in the x-axis:

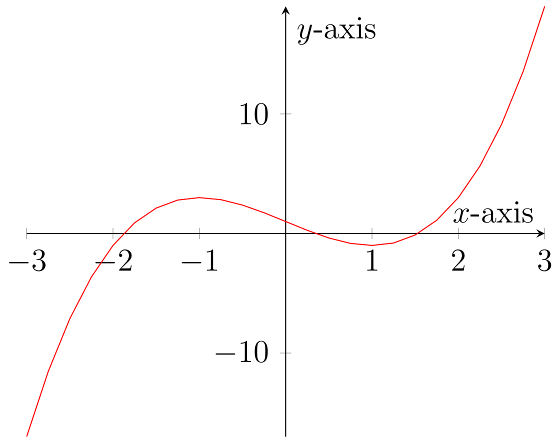


The graph of $f(kx)$.

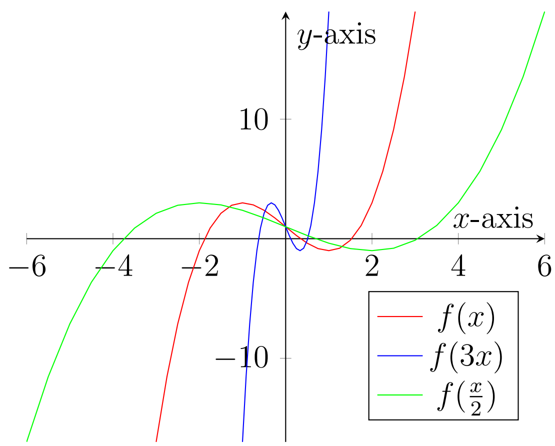
If $k > 1$ the graph of $f(kx)$ is obtained from the graph of $f(x)$ by shrinking the x-axis k times.

If $0 < k < 1$ the graph of $f(kx)$ is obtained from the graph of $f(x)$ expanding the x-axis $\frac{1}{k}$ times.

For example, if the graph of $f(x)$ is given below

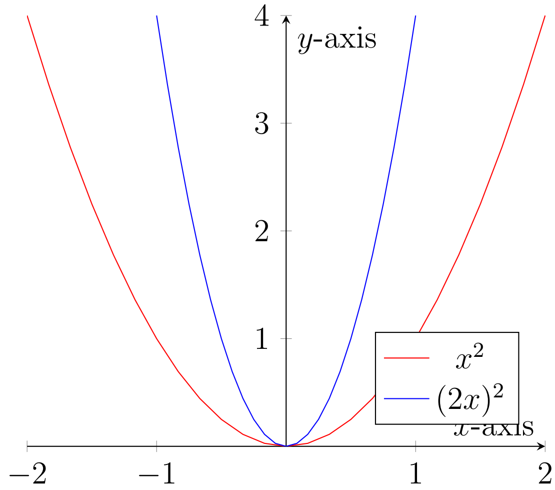


then, the graphs of $f(3x)$ and $f(\frac{x}{2})$ are



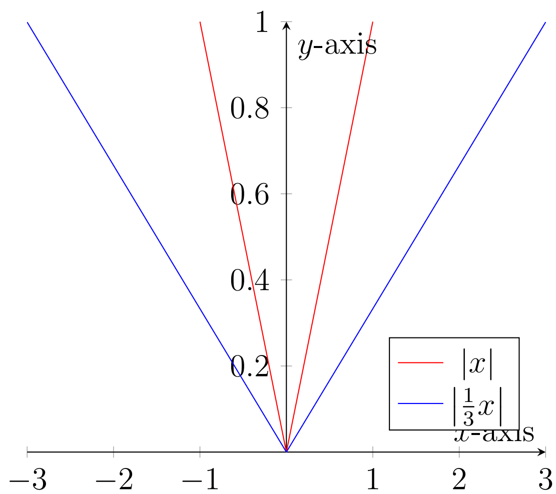
Exercise 2.9.9 *Sketch the graph of* $(2x)^2$.

If $f(x) = x^2$ we have $f(2x) = (2x)^2$. Thus, we start with the graph of $f(x) = x^2$ and we shrink the x-axis by a factor of 2:



Exercise 2.9.10 *Sketch the graph of* $\left|\frac{1}{3}x\right|$.

If $f(x) = |x|$ we have $f(\frac{1}{3}x) = \left|\frac{1}{3}x\right|$. Thus, we start with the graph of $f(x) = |x|$ and we expand the x-axis three times:

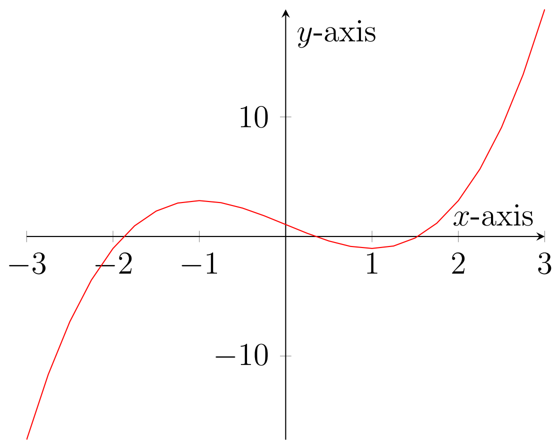


The graph of $kf(x)$.

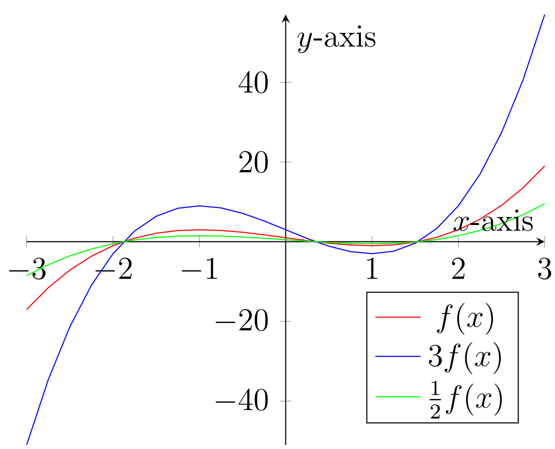
If $k > 1$ the graph of $kf(x)$ is obtained from the graph of $f(x)$ by expanding the y-axis k times.

If $0 < k < 1$ the graph of $f(kx)$ is obtained from the graph of $f(x)$ shrinking the y-axis $\frac{1}{k}$ times.

For example, if the graph of $f(x)$ is given below

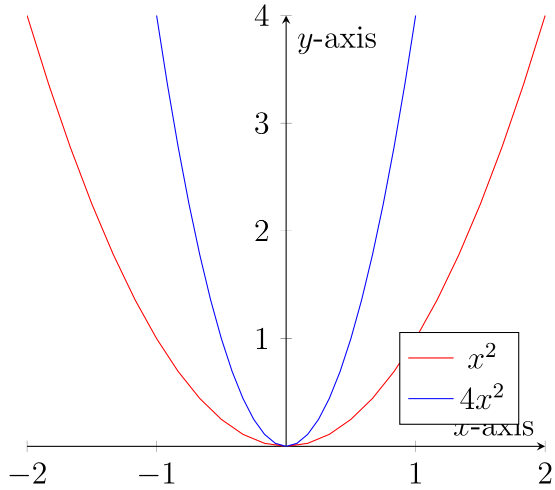


then, the graphs of $3f(x)$ and $\frac{1}{2}f(x)$ are



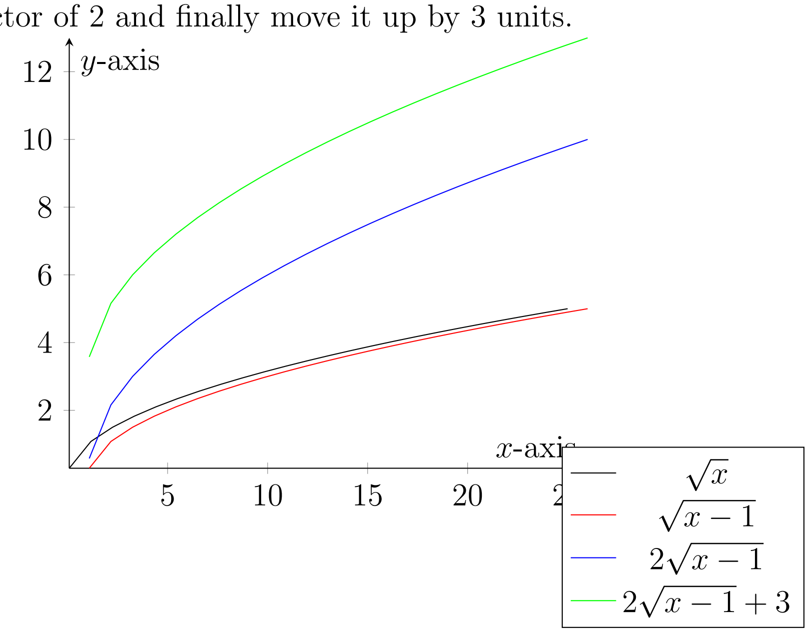
Exercise 2.9.11 *Sketch the graph of* $4x^2$.

If $f(x) = x^2$ we have $4f(x) = 4x^2$. Thus, we start with the graph of $f(x) = x^2$ and we shrink the x-axis by a factor of 4:



Exercise 2.9.12 *Sketch the graph of* $2\sqrt{x-1}+3$.

We start at $\sqrt{x}$ move it one unit to the left, then expand the y axis by a factor of 2 and finally move it up by 3 units.



Note: The order in which we do the operations matters. We always start from x and work our way out. This is the same order we would calculate if we replace x by a number.

2.9.7 Drawing quadratic functions

By completing the square we can draw the graph of any function of the form $f(x) = ax^2 + bx + c$. Indeed, by completing the square, we can write f in the form

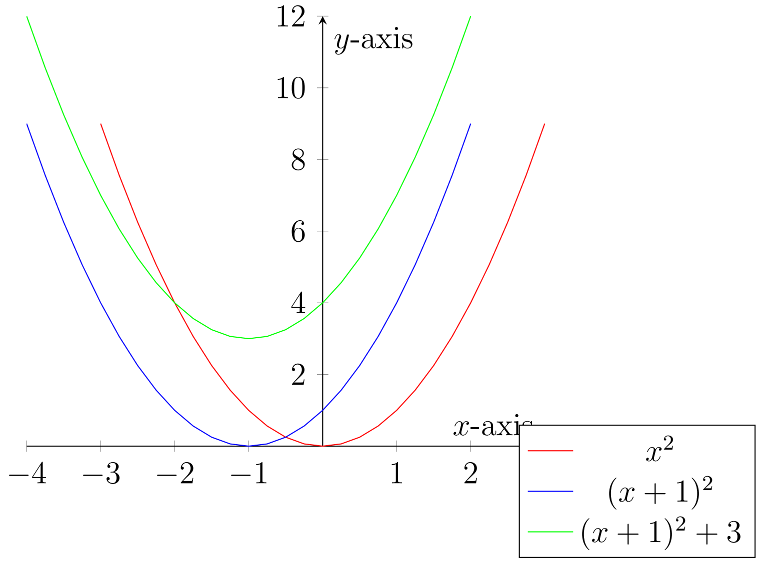
$$f(x) = a(x + e)^2 + f\,.$$

Therefore, the graph of f can be drawn by staring with the graph of x^2, moving it e units to left/right, then expanding/contracting the y axis by a factor of a and then moving up/down by f units.

Exercise 2.9.13 *Sketch the graph of* $x^2 + 2x + 4$.

Completing the square we get

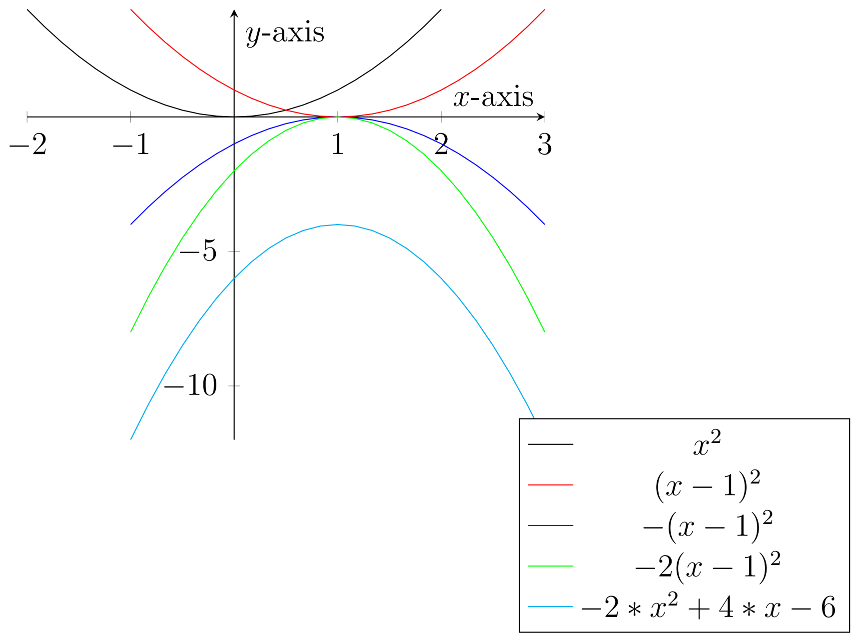
$$x^2 + 2x + 4 = x^2 + 2x + 1 - 1 + 4 = (x + 1)^2 + 3$$



Exercise 2.9.14 *Sketch the graph of* $-2x^2 + 4x - 6$.

Completing the square we get

$$\begin{aligned}-2x^2 + 4x - 6 &= -2\left(x^2 - 2x + 3\right) = -2\left(x^2 - 2x + 1 - 1 + 3\right)\\ &= -2\left((x - 1)^2 + 2\right) = -2(x - 1)^2 - 4\end{aligned}$$

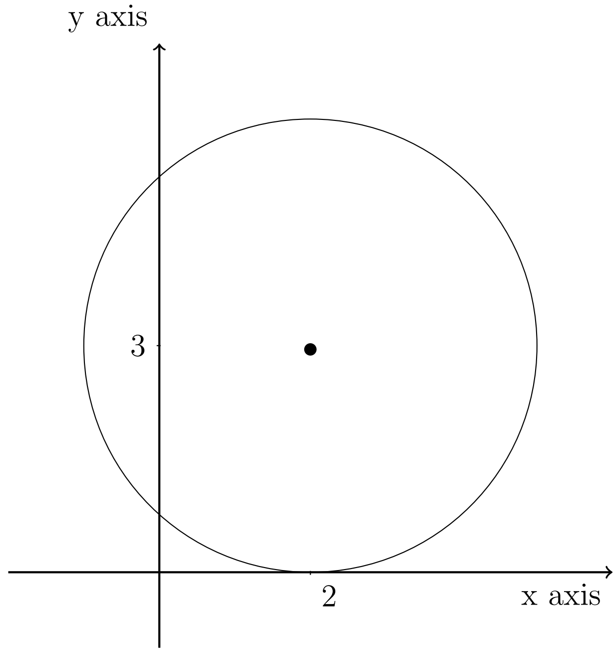


2.9.8 Circles

The circle with center (a,b) and radius R is the curve given by the equation

$$(x-a)^2+(y-b)^2=R^2\,.$$

Example 2.9.1 *Sketch the circle* $(x-2)^2+(y-3)^2=3^2$.



Exercise 2.9.15 *Find the center and the radius of*

$$(x-1)^2 + (y+2)^2 = 4$$

Answer: Center is $(1, -2)$ and radius is 2.

Exercise 2.9.16 *Find the center and the radius of*

$$x^2 + (y+1)^2 = 7$$

Answer: Center is $(0, -1)$ and radius is $\sqrt{7}$.

Exercise 2.9.17 *Write the equation of the circle with the center* $(-1, 2)$ *and radius* 3.

Answer:

$$(x+1)^2 + (y-2)^2 = 9\,.$$

Exercise 2.9.18 *Draw*

$$x^2 + 2x + y^2 - 6y = 6\,.$$

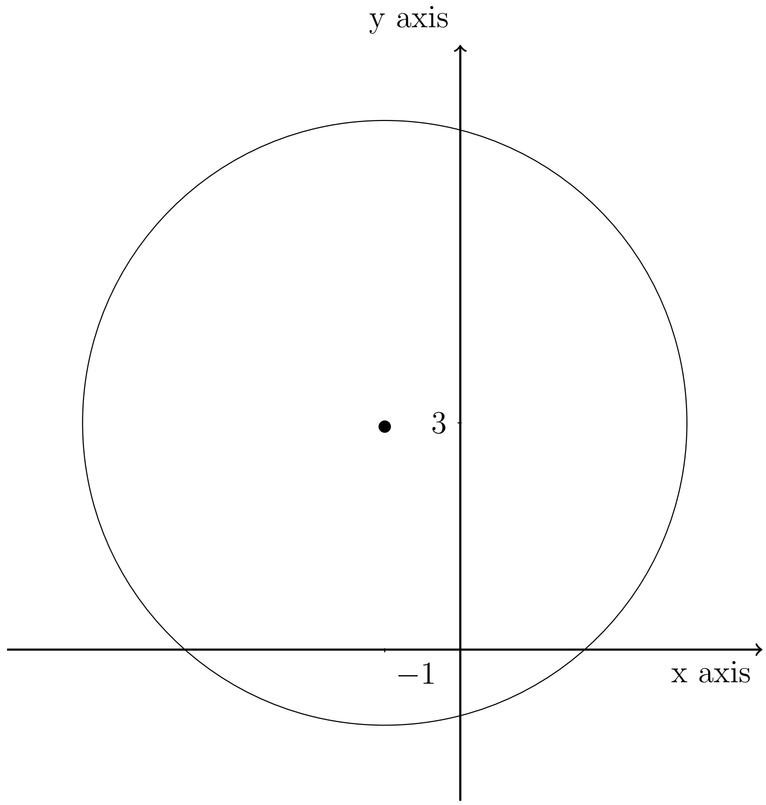
Sol: Completing the square we get

$$x^2 + 2x + 1 - 1 + y^2 - 6y + 9 - 9 = 6\,.$$

$$(x + 1)^2 + (y - 3)^2 - 10 = 6\,.$$

$$(x + 1)^2 + (y - 3)^2 = 16\,.$$

The circle has center $(-1, 3)$ and radius 4.



2.9.9 Exercises

Exercise 2.9.19 *Draw the following graphs:*

i) $y = \frac{1}{x-2}$.

ii) $y = x^2 - 4x + 1$

iii) $y = -2x^2 - 4x + 6$.

Exercise 2.9.20 *i) Find an equation for the circle of radius* 3 *with centre* $(2, -1)$.

ii) Find the center and radius of the circle

$$X^2 - 4X + Y^2 + 2Y = 4\,.$$

Exercise 2.9.21 *Sketch the graph of* $y = x^2 - 4x + 3$.

Exercise 2.9.22 *Find the center and radius of the circle*

$$X^2 + 4X + Y^2 - 6Y = 3\,.$$

Exercise 2.9.23 *Sketch the graph of* $y = x^2 - 2x - 3$.

Exercise 2.9.24 *Find the center and radius of the circle*

$$X^2 + 2X + Y^2 - 6Y = 15\,.$$

Exercise 2.9.25 *Sketch the graph of* $y = x^2 + 4x - 5$.

Exercise 2.9.26 *i) Write an equation for the circle with center* $(1, 2)$ *and radius* 3.

ii) Find the center and radius of the circle

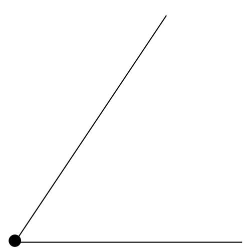
$$X^2 - 2X + Y^2 + 4Y = 11\,.$$

Chapter 3

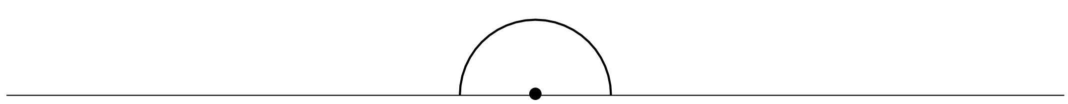
Introduction to Geometry

3.1 Angles

Definition 3.1.1 *An* **angle** *is the figure made by two rays at a common point. We use the symbol* $\angle$ *to denote an angle.*



An angle made by two rays which make a line is considered to have 180°:

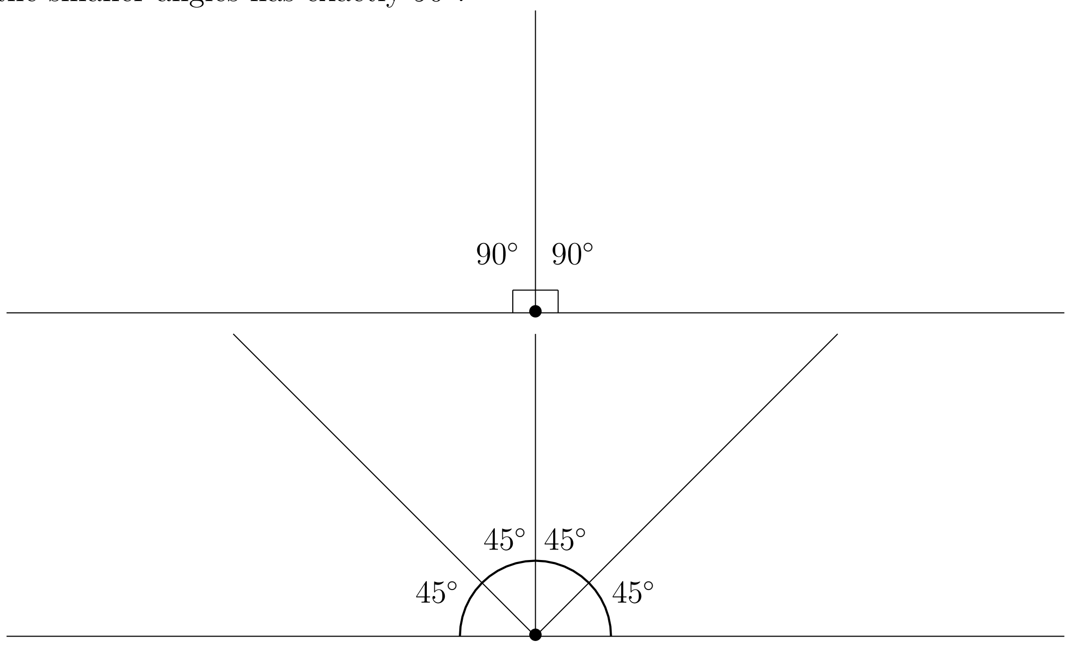


Definition 3.1.2 *Two angles are called* **congruent** *if we can translate and rotate one angle in such a way that it completely overlaps with another angle.*

We use the symbol $\equiv$ *for congruence.*

If we split an angle of d degrees in n congruent angles, each of the smaller angle has exactly $\frac{d}{n}$ degrees.

For example, if we split an angle of 180° into two congruent angles, each of the smaller angles has exactly 90°.



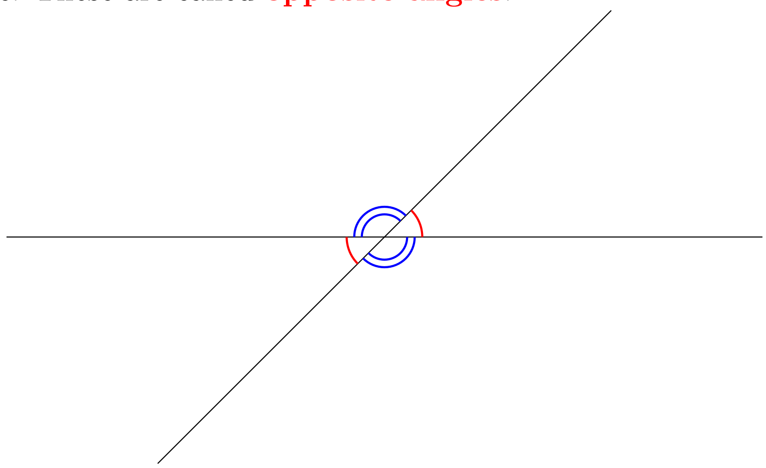
The measures of angles can tell us if angles are congruent or not:

Theorem 3.1.1 *Two angles are congruent if and only if they have the same measure.*

An angle is called **right angle** if it measures 90°.
An angle which measures less that 90° is called an **acute angle**.
An angle which measures more that 90° is called an **obtuse angle**.
Two angles are called **supplementary** their measures add to 180°.

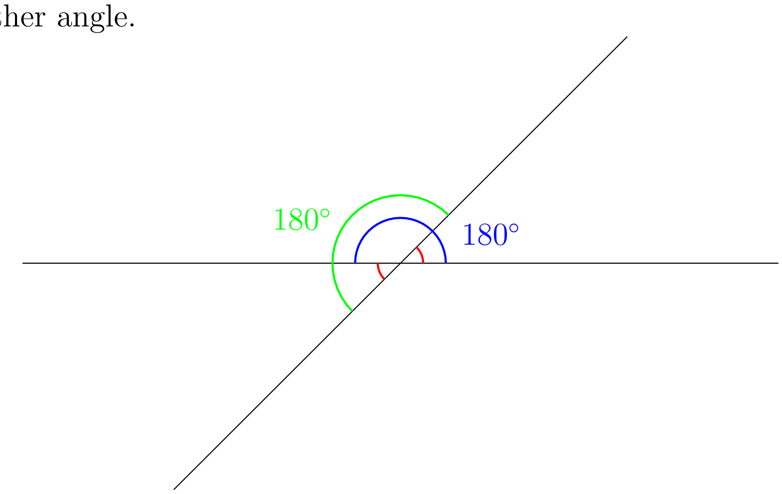
3.1.1 Opposite, alternating interior and corresponding angles

If two lines intersect at a point they make two pairs of angles on opposite side. These are called **opposite angles**.



Theorem 3.1.2 *Opposite angles are congruent.*

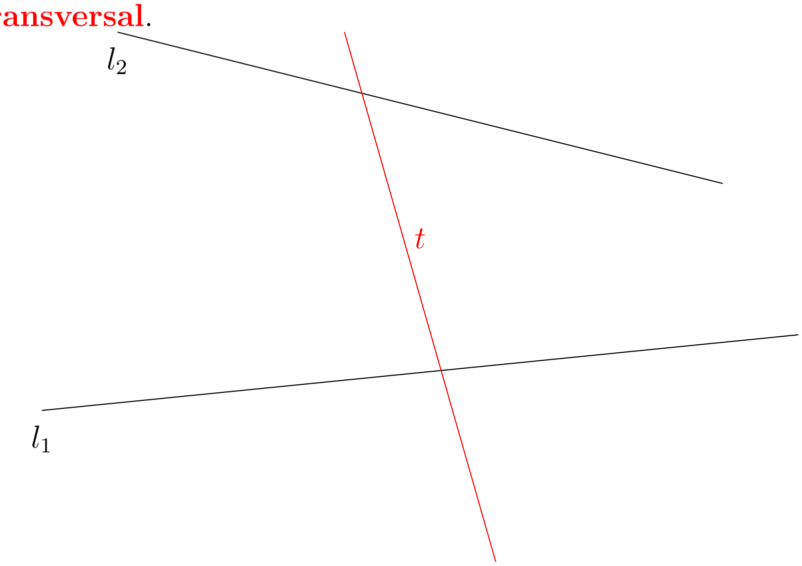
Proof: If we rotate one of the angles by 180° it completely overlaps with the other angle.



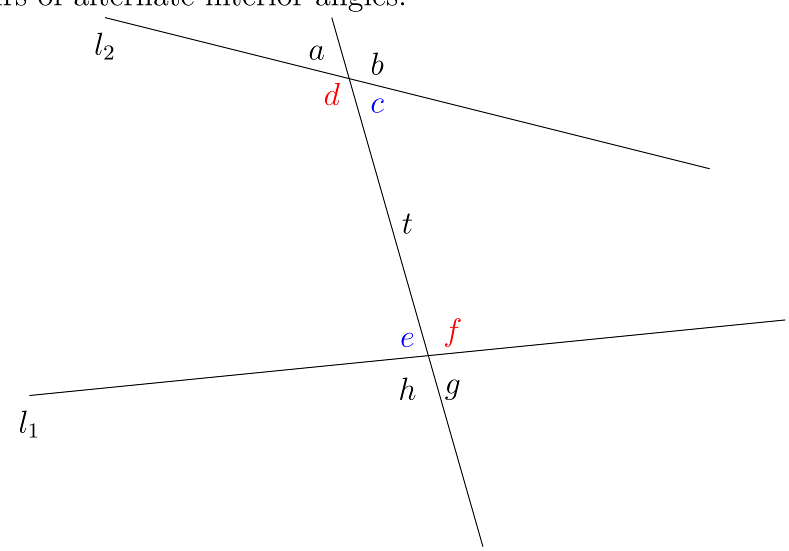
Alternately, we can observe that we can add the same angle to both of the given angles to get 180°.

□

Given two lines l_1, l_2, a line t which intersects both of them is called **a transversal**.

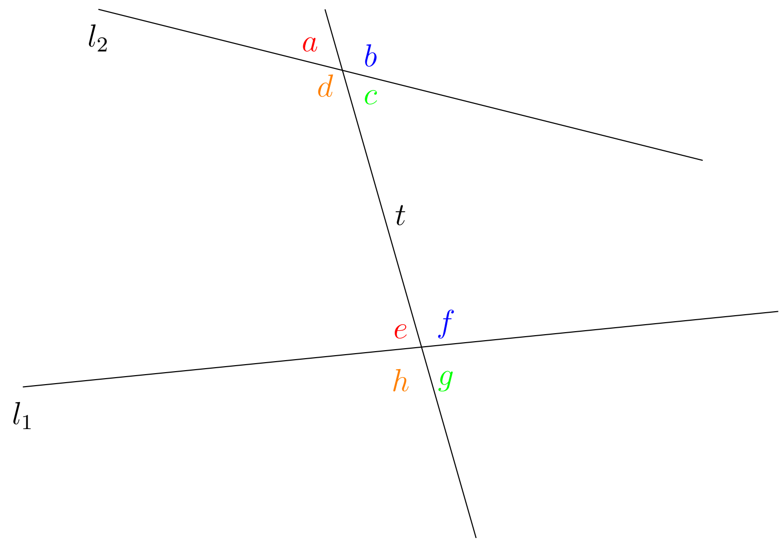


Two angles on opposite side of t, in between l_1, l_2, one made by l_1 and t, the other by l_2 and t are called **alternate interior angles**. There are two pairs of alternate interior angles:



In this picture, the angles c and e are alternate interior angles. The angles d and f are also alternate interior angles.

if two lines l_1, l_2 are cut by a transversal, angles in matching corners are called **corresponding**. There are four pairs of corresponding angles:



In this picture, the angles a and e are corresponding angles. The angles b and f are corresponding angles. The angles c and g are corresponding angles. The angles d and h are corresponding angles.

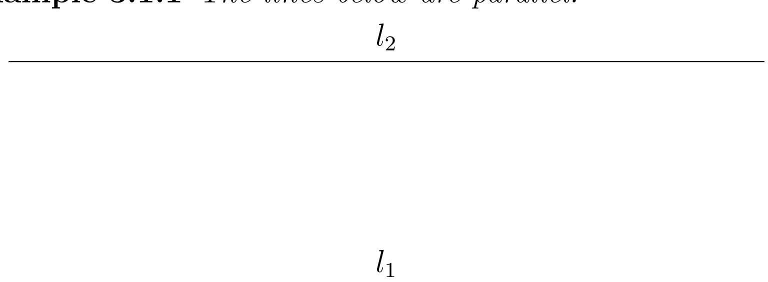
Let us recall the definition of parallel lines:

Definition 3.1.3 *Two lines l_1 and l_2 are called* **parallel** *if they don't intersect. We use the notation*

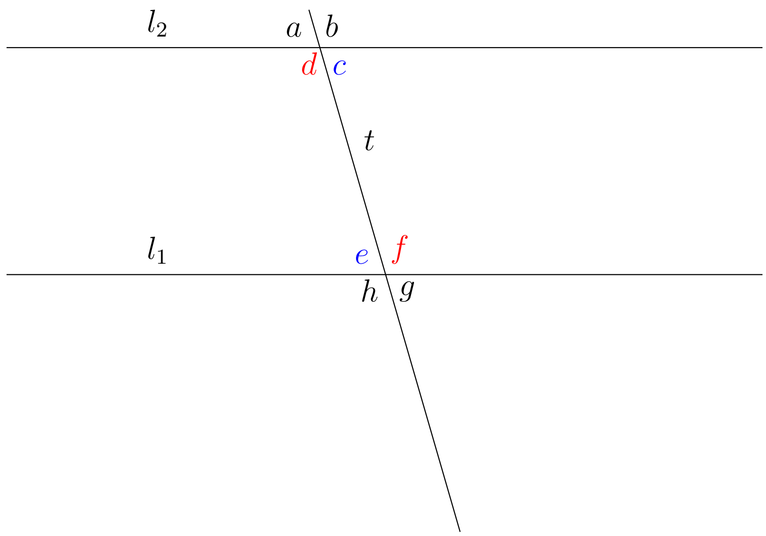
$$l_1 \parallel l_2\,,$$

to say that the lines are parallel.

Example 3.1.1 *The lines below are parallel:*



Theorem 3.1.3 *If two parallel lines are cut by a transversal t, the alternate interior angles are congruent.*

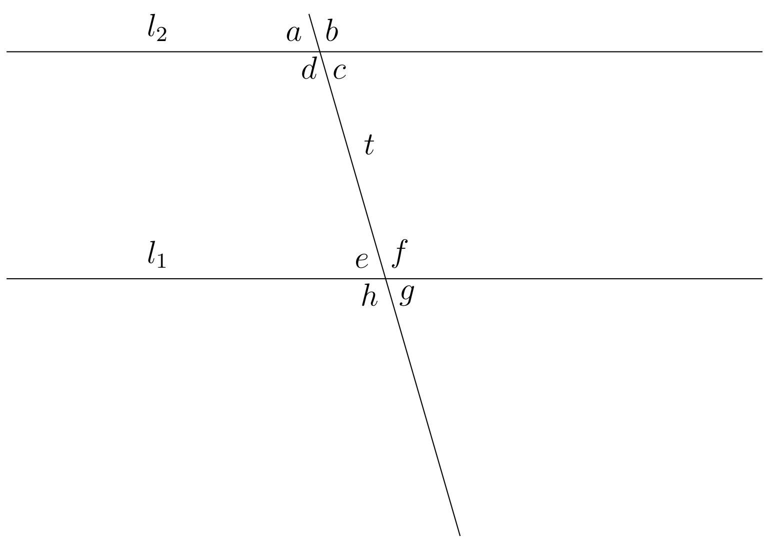


The theorem tells us that if we are given that $l_1 \parallel l_2$, then we know that

$$\angle d \equiv \angle f \qquad \text{and}$$
$$\angle c \equiv \angle e$$

The following is a consequence of the Theorem3.1.3:

Theorem 3.1.4 *If two parallel lines are cut by a transversal t, then the corresponding angles are congruent.*



Proof:
By Theorem 3.1.3 we have

$$\angle d \equiv \angle f \qquad \text{and}$$
$$\angle c \equiv \angle e$$

As opposite angles we also have

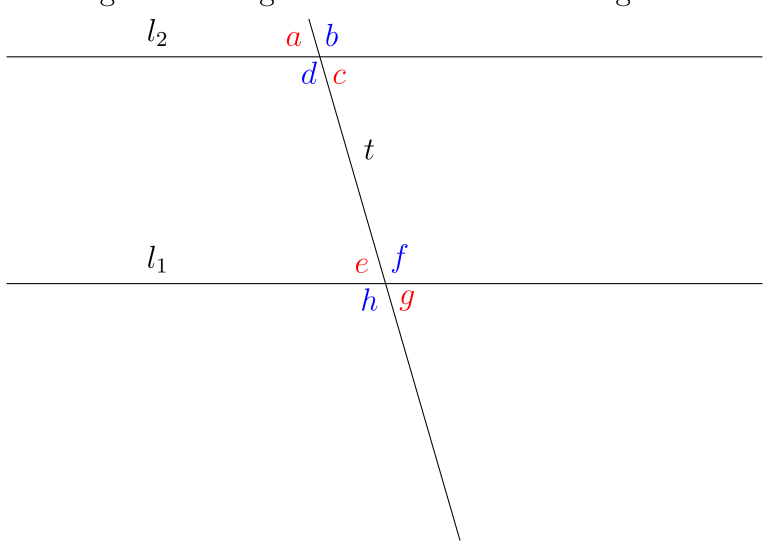
$$\angle a \equiv \angle c$$
$$\angle b \equiv \angle d$$
$$\angle e \equiv \angle g$$
$$\angle f \equiv \angle h$$

Combining the two relations, we get

$$\angle a \equiv \angle c \quad \equiv \angle e \equiv \angle g$$
$$\angle b \equiv \angle d \quad \equiv \angle f \equiv \angle h$$

□

It follows that given two parallel lines $l_1 \parallel l_2$ cut by a transversal t, all the red angles are congruent and all the blue angles are congruent:



The converses of the previous results are also true:

Theorem 3.1.5 *If two lines are l_1, l_2 are cut by a transversal t, and one pair of alternate interior angles are congruent, then the two lines are parallel.*

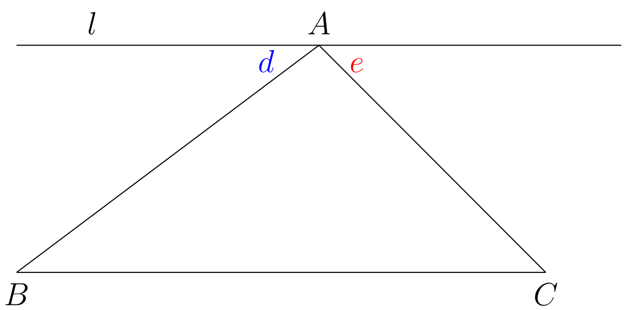
Theorem 3.1.6 *If two lines are l_1, l_2 are cut by a transversal t, and one pair of corresponding angles are congruent, then the two lines are parallel.*

3.1.2 Sum of angles in a triangle

Theorem 3.1.7 *The sum of angles in any triangle is $180°$.*

Proof: We will denote the triangle by ABC and we will call the three angles in the triangle $\angle A, \angle B, \angle C$.

We draw a line l parallel through A to the line BC. We denote the two extra angles we get by $\angle d$ and $\angle e$, as on the picture below:



The parallel lines $l \parallel BC$ are cut by the transversal AB. Therefore, by Theorem 3.1.3 we have

$$\angle B \equiv \angle d\,.$$

The parallel lines $l \parallel BC$ are cut by the transversal AC. Therefore, by Theorem 3.1.3 we have

$$\angle C \equiv \angle e\,.$$

It follows that

$$\angle A + \angle B + \angle C = \angle A + \angle d + \angle e = 180°\,.$$

□

Exercise 3.1.1 *In a triangle ABC we know that $\angle A = 60°$ and $\angle B = 50°$. Find $\angle C$.*

Solution:

$$180^\circ = \angle A + \angle B + \angle C = 60^\circ + 50^\circ + \angle C\,.$$

It follows that

$$\angle C = 70^\circ\,.$$

Exercise 3.1.2 *In a triangle ABC we know that $\angle A = 90^\circ$. What is $\angle B + \angle C$.*

Solution:

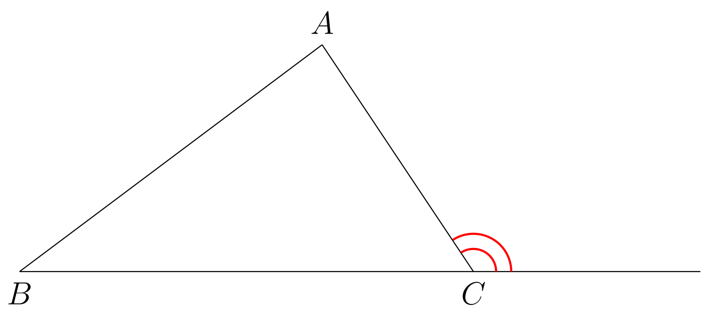
$$180^\circ = \angle A + \angle B + \angle C = 90^\circ + \angle B + \angle C\,.$$

It follows that

$$\angle B + \angle C = 90^\circ\,.$$

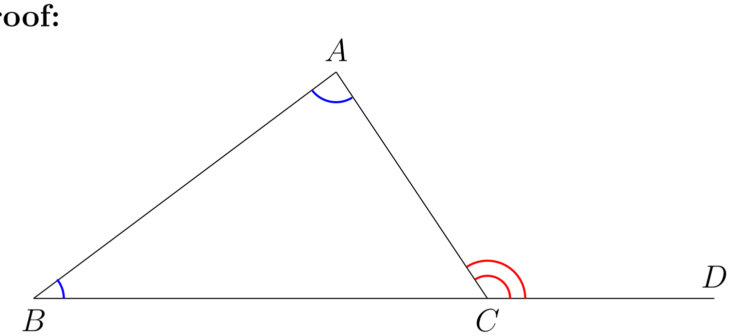
Definition 3.1.4 *In a triangle, the angle made by one side with the extension of another side is called an* **exterior angle***.*

The angle emphasized below is exterior angle C:



Theorem 3.1.8 *Any exterior angle in a triangle is equal to the sum of the other two angles.*

Proof:



We know that

$$\angle A + \angle B + \angle ACB = 180^\circ = \angle ACB + \angle DCB \, .$$

Canceling $\angle ACB$ we get

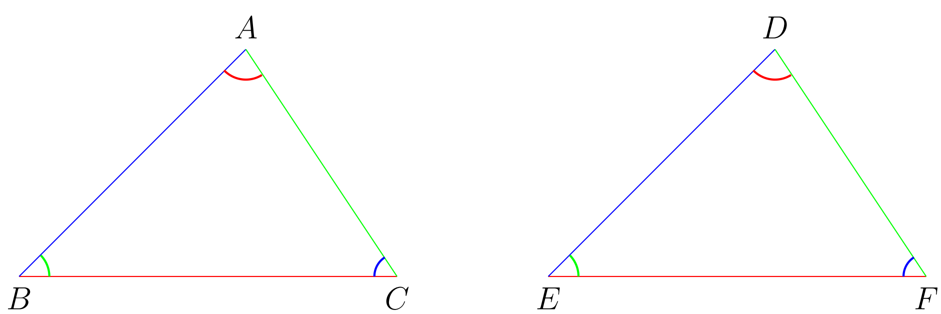
$$\angle A + \angle B = \angle DCB \, .$$

□

3.2 Congruent Triangles

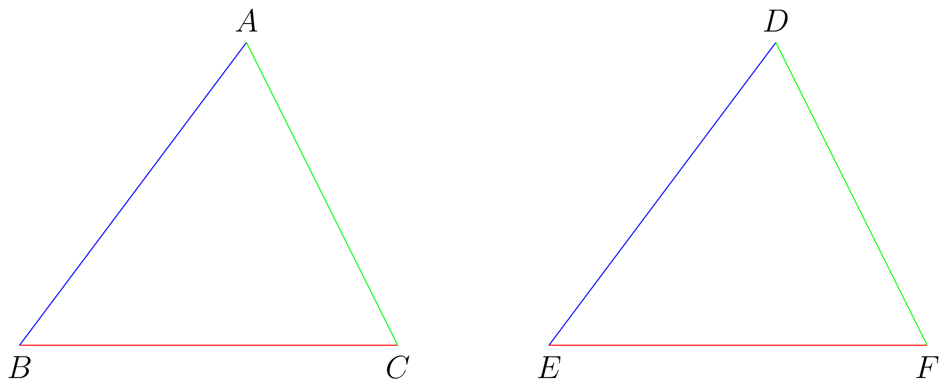
Definition 3.2.1 *Two triangles are called* **congruent** *if they have corresponding equal sides and equal angles.*

For example the two triangles below are congruent:



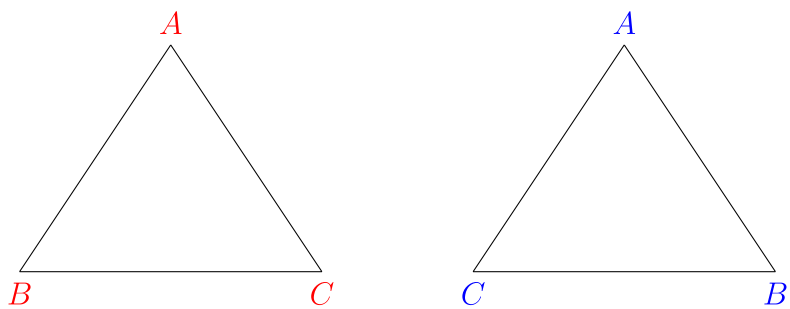
It turns out that in general we don't need to check all six. In the following cases, by checking (the right) three of them we get for free the remaining three.

Theorem 3.2.1 *(**SSS**) If two triangles have equal side lengths, they are congruent.*



Exercise 3.2.1 *Let ABC be a triangle. If $AB \equiv AC$ then show that $\angle B \equiv \angle C$.*

Solution: We compare ABC with it mirror reflection ACB.



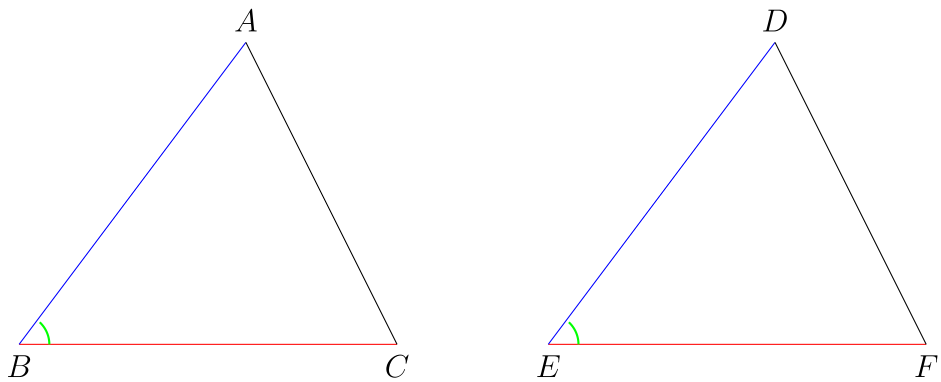
$$AB \equiv AC$$
$$AC \equiv AB$$
$$BC \equiv CB$$

Therefore, by **SSS** we have

$$\Delta ABC \equiv \Delta ACB\,.$$

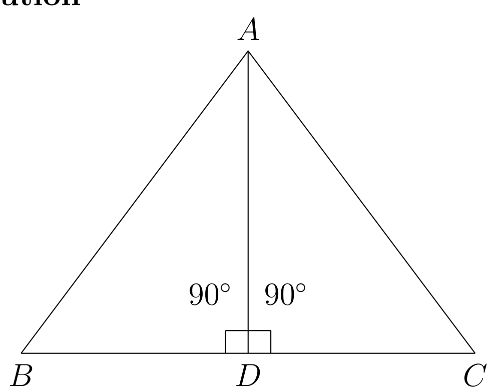
This implies that $\angle B \equiv \angle C$.

Theorem 3.2.2 *(**SAS**) If two triangles have two pairs of equal sides, and if the angles between those sides are also congruent, then the triangles are congruent.*



Exercise 3.2.2 *Let ABC be a triangle and let D be a point on side BC such that $AD \perp BC$. If $BD \equiv DC$ prove that $AB \equiv AC$.*

Solution



We have

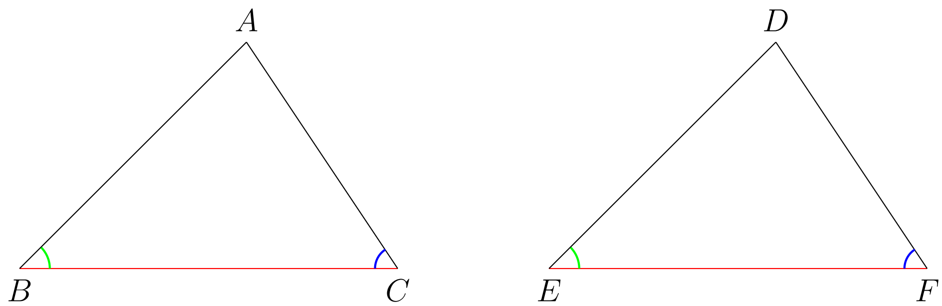
$$AD \equiv AD$$
$$BDCD$$
$$\angle BDA \equiv \angle CDA$$

Therefore, by **SAS** we have

$$\Delta BDA \equiv \Delta CDA\,.$$

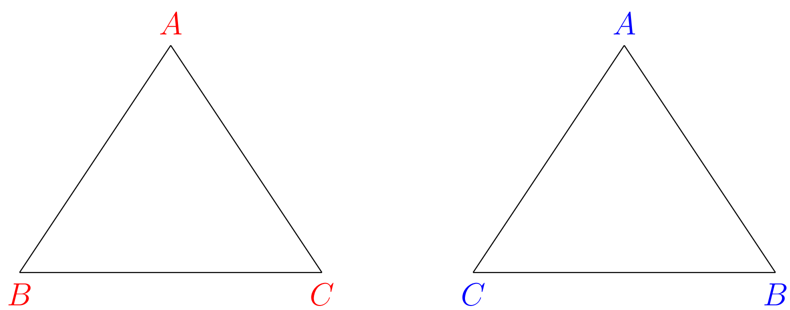
This implies that $AB \equiv AC$.
Note: Exercise 3.2.1 can also be solved with **SAS**.

Theorem 3.2.3 *(**ASA**) If two triangles have two pairs of equal angles, and if the sides between those angles are also congruent, then the triangles are congruent.*



Exercise 3.2.3 *Let ABC be a triangle. If $\angle B \equiv \angle C$ then show that $AB \equiv AC$.*

Solution: We compare ABC with it mirror reflection ACB.



$$\angle B \equiv \angle C$$
$$BC \equiv CB$$
$$\angle C \equiv \angle B$$

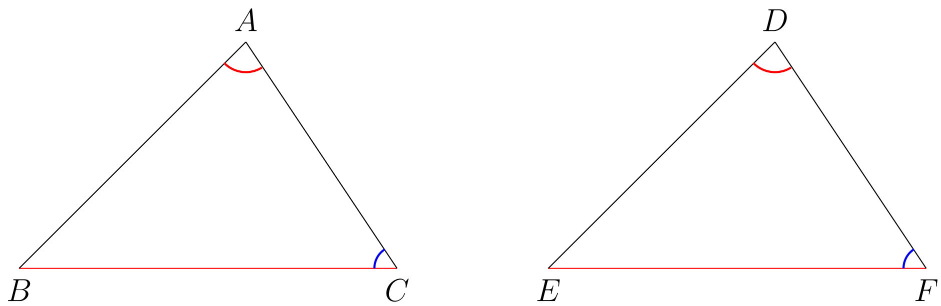
Therefore, by **ASA** we have

$$\Delta ABC \equiv \Delta ACB\,.$$

This implies that $AB \equiv AC$.

By using the sum of the angles in a triangle, we get that the case AAS also works.

Theorem 3.2.4 *(**AAS**) If two triangles have two pairs of equal angles, and if a pair of corresponding sides not between those angles are also congruent, then the triangles are congruent.*



Important: The cases AAA and SSA don't work. If two triangles are in one of those two cases, it doesn't mean that they are congruent.

3.2.1 Isosceles and Equilateral Triangles

Definition 3.2.2 *A triangle which has two equal sides is called* **isosceles**. *A triangle which has all sides equal is called* **equilateral**.

In Exercise 3.2.1 we proved the following Theorem.

Theorem 3.2.5 *In an isosceles triangle, the angles opposing congruent sides are congruent.*

In Exercise 3.2.3 we proved that the converse of this Theorem is also true:

Theorem 3.2.6 *If two angles in a triangle are congruent, then the triangle is isosceles.*

In the case of equilateral triangles, by using these theorems twice we get:

Theorem 3.2.7 *In an equilateral triangle, all angles are congruent.*

Theorem 3.2.8 *If all angles in a triangle are congruent, then the triangle is equilateral.*

Exercise 3.2.4 *Find the angles of an equilateral triangle.*

Solution: We will denote the triangle by ABC. Since the triangle is equilateral, we know that $\angle A \equiv \angle B \equiv \angle C$.

Therefore

$$180^\circ = \angle A + \angle B + \angle C = 3\angle A\,.$$

It follows that

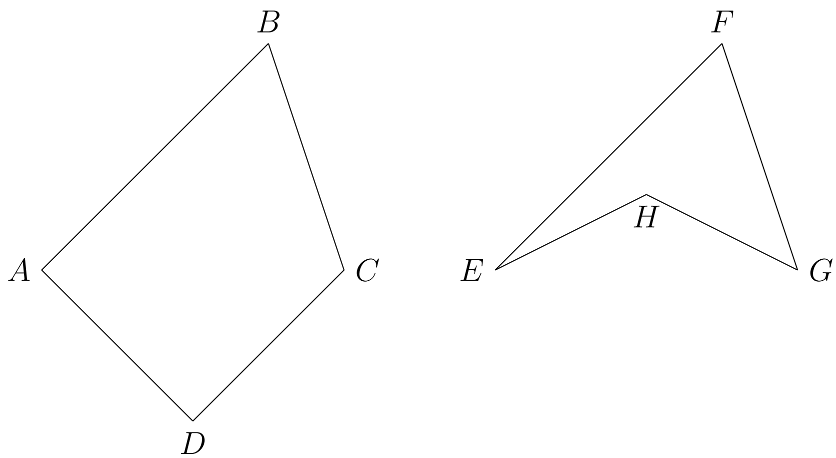
$$\angle A = \angle B = \angle C = 60^\circ\,.$$

3.3 Quadrilaterals

Definition 3.3.1 *A figure with four sides and 4 vertices is called a* **quadrilateral**.

If the quadrilateral has no interior angle larger than 180° *is called* **convex**. *A quadrilateral with an angle larger than* 180° *is called* **concave**.

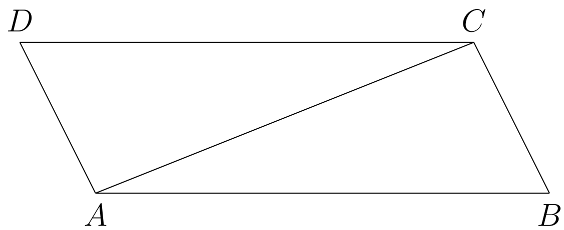
Example 3.3.1 *In the picture below,*



$ABCD$ is a convex quadrilateral, while $EFGH$ is a concave quadrilateral.

Exercise 3.3.1 *Let $ABCD$ let be a quadrilateral such that $AB \parallel CD$ and $AD \parallel BC$. Prove that $AB \equiv CD$ and $AD \equiv BC$.*

We draw the diagonal AC.



The parallel lines $AB \parallel CD$ are cut by the transversal AC. It follows that the alternate interior angles are congruent:

$$\angle CAB \equiv \angle DCA\,.$$

The parallel lines $AD \parallel BC$ are cut by the transversal AC. It follows that the alternate interior angles are congruent:

$$\angle ACB \equiv \angle CAD\,.$$

Therefore we have:

$$\begin{aligned} \angle CAB &\equiv \angle DCA \\ AC &\equiv AC \\ \angle ACB &\equiv \angle CAD \end{aligned}$$

Therefore, by **ASA** we have

$$\Delta ABC \equiv \Delta CDA\,.$$

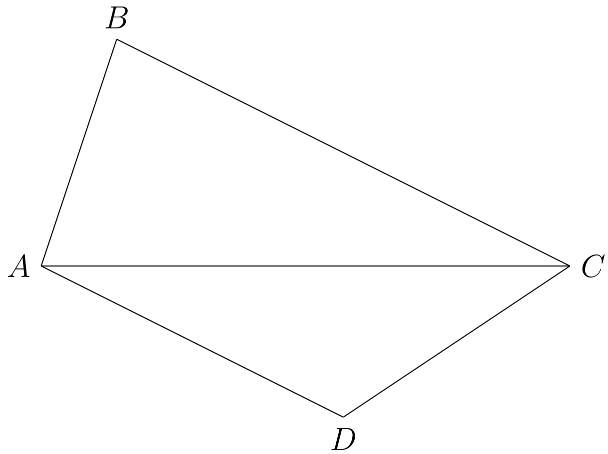
From here we get:

$$\begin{aligned} AB &\equiv CD \\ AD &\equiv BC \end{aligned}$$

Definition 3.3.2 *A quadrilateral in which the opposite edges are parallel is called a* **parallelogram**.

Exercise 3.3.2 *Let $ABCD$ let be a convex quadrilateral. Find the sum of the four angles.*

We draw the diagonal AC.



The sum of angles in ΔABC is $180°$:

$$\angle BAC + \angle B + \angle BCA = 180^\circ\,.$$

The sum of angles in ΔADC is $180°$:

$$\angle DAC + \angle D + \angle DCA = 180^\circ\,.$$

Adding together these relations we get

$$\angle BAC + \angle B + \angle BCA + \angle DAC + \angle D + \angle DCA = 360^\circ\,.$$

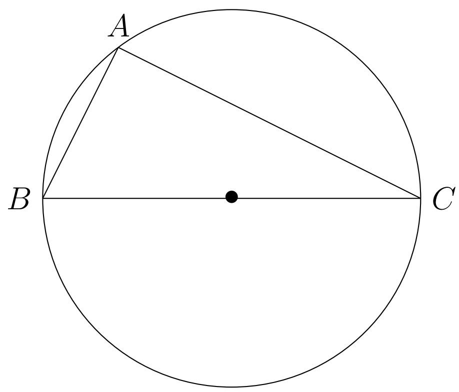
Therefore

$$\angle A + \angle B + \angle C + \angle D = 360^\circ\,.$$

3.4 Thales Theorem

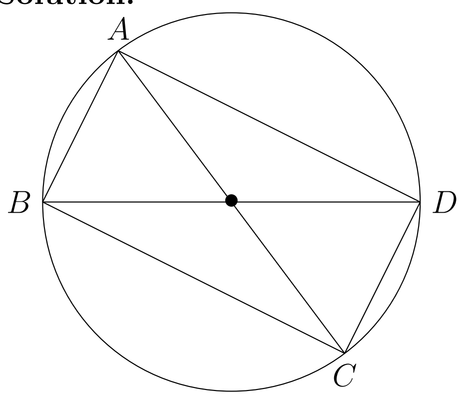
Theorem 3.4.1 *(**Thales**) Let A, B, C be three points on a circle such that BC is a diameter. Then*

$$\angle BAC = 90^\circ\,.$$



Exercise 3.4.1 *Let A, B, C, D be four points on a circle such that AC and BD are diameters. Show that $ABCD$ is a rectangle.*

Solution:



Since AC is a diameter, by **Thales** Theorem $\angle B = \angle D = 90°$.
Since BD is a diameter, by **Thales** Theorem $\angle A = \angle C = 90°$.
Therefore

$$\angle A = \angle B = \angle C = \angle D = 90° \,.$$

This shows that $ABCD$ is a rectangle.

3.5 Areas

The area is a quantity which measures how big a figure is in the plane. The basic property of area is the following Theorems:

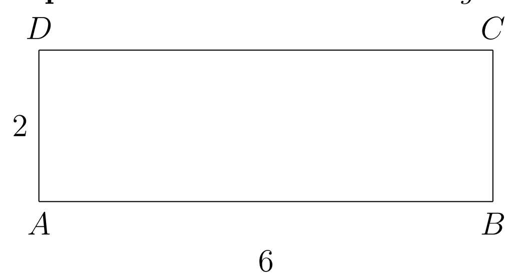
Theorem 3.5.1 *If we cut a shape into pieces, the area of the shape is equal to the sum of areas of pieces.*

Theorem 3.5.2 *Congruent pieces have equal areas.*

3.5.1 Area of a rectangle

Definition 3.5.1 *The area of a rectangle is equal to the length of the base times the length of the height.*

Example 3.5.1 *Consider the rectangle $ABCD$ below:*

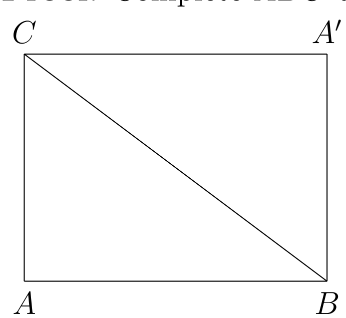


Its area is $2 \cdot 6 = 12$ *square units.*

3.5.2 Area of a triangle

Theorem 3.5.3 *Let ABC be a triangle with $\angle A = 90°$. Then the area of ABC is $\frac{AB \cdot AC}{2}$.*

Proof: Complete ABC to a rectangle.



As $\Delta ABC \equiv \Delta A'BC$, it follows that the area of ABC is half of the area of $ABA'C$. As the area of the rectangle $ABA'C$ is $AB \cdot AC$, we obtain the desired result.

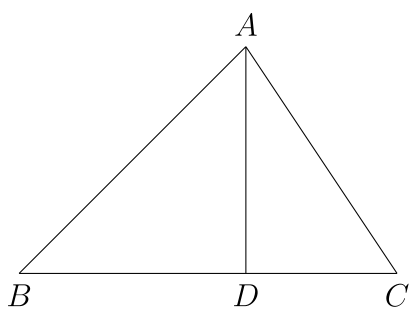
□

Exercise 3.5.1 *ABC is a triangle with $\angle A = 90°$ and $AB = 3, AC = 4$. Find its area.*

Solution Area is $\frac{3 \cdot 4}{2} = 6$.

Theorem 3.5.4 *The area of a triangle is base times hight divided by 2.*

Proof: Consider a triangle ABC and let D be a point on the side BC so that $AD \perp BC$.



Then, the area of the right triangle ABD is $\frac{AD \cdot BD}{2}$. The area of the right triangle ACD is $\frac{AD \cdot CD}{2}$.

Adding them together we get that the area of ABC is

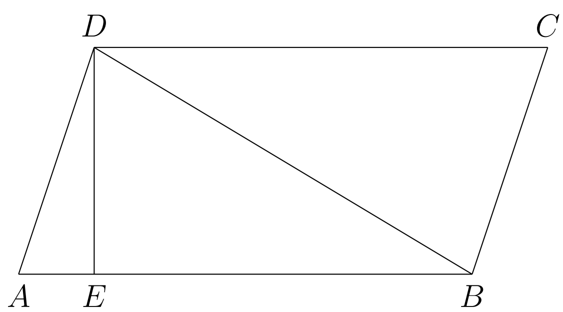
$$\frac{AD \cdot BD}{2} + \frac{AD \cdot CD}{2} = \frac{AD \cdot (BD + CD)}{2} = \frac{AD \cdot BC}{2}.$$

□

3.5.3 Area of a parallelogram

Theorem 3.5.5 *The area of a parallelogram is base times heigth.*

Proof: Consider a parallelogram $ABCD$ and let AE be its height. Draw the diagonal BD.



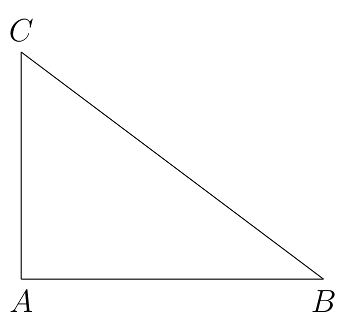
Then, the triangles ABD and CBD are congruent. Therefore, the area of the parallelogram is twice the area of ABD.

As the area of ABD is $\frac{AB \cdot DE}{2}$, the claim follows.

3.6 Pythagoras' Theorem

Theorem 3.6.1 *Let ABC be a triangle with $\angle A = 90^\circ$. Then*

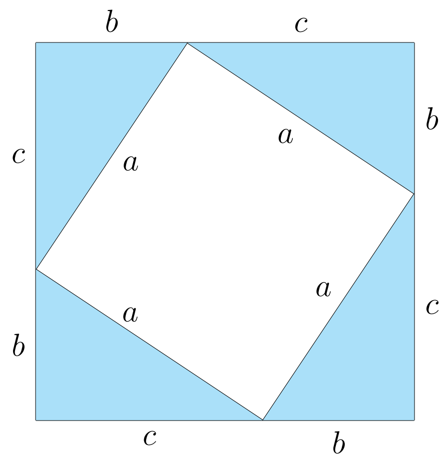
$$AB^2 + AC^2 = BC^2 .$$



Proof: We give a nice proof of Pytagora's Theorem, based on areas. For simplicity we will denote $AB = c, AC = B, BC = a$. We need to prove that

$$a^2 + b^2 = c^2 .$$

Add together four copies of the triangle ABC as in the picture below:



The outside shape is a square with side $b+c$, therefore the total area in the picture is $(b+c)^2 = b^2 + c^2 + 2bc$.

The inside shape is a square with side a, therefore the white area is a^2.

Each of the blue triangles is a right triangle of area $\frac{b \cdot c}{2}$. Therefore the blue area is $4 \cdot \frac{b \cdot c}{2} = 2bc$.

We know that the total area is equal to the white area plus the blue area. Therefore

$$b^2 + c^2 + 2bc = a^2 + 2bc\,.$$

Canceling $2bc$ from both sides we get the Pytagorean theorem.

□

Exercise 3.6.1 *ABC is a triangle with $\angle A = 90^\circ$. If $AB = 3, AC = 4$ find BC.*

Solution:

By Pytagora's Theorem

$$BC^2 = AB^2 + AC^2 = 3^2 + 4^2 = 9 + 16 = 25\,.$$

Therefore $BC = 5$.

Exercise 3.6.2 *ABC is a triangle with $\angle A = 90^\circ$. If $AB = 2, AC = 3$ find BC.*

Solution:

By Pytagora's Theorem

$$BC^2 = AB^2 + AC^2 = 2^2 + 3^2 = 4 + 9 = 13\,.$$

Therefore $BC = \sqrt{13}$.

Exercise 3.6.3 *ABC is a triangle with $\angle A = 90°$. If $AB = 5, BC = 13$ find AC.*

Solution:

By Pytagora's Theorem

$$13^2 = BC^2 = AB^2 + AC^2 = 5^2 + AC^2 = 25 + AC^2\,.$$

Thus

$$AC^2 = 169 - 25 = 144\,.$$

Therefore $AC = \sqrt{144} = 12$.

Exercise 3.6.4 *ABC is a triangle with $\angle A = 90°$. If $AB = 3, BC = 7$ find the area of ABC.*

Solution:

By Pytagora's Theorem

$$7^2 = BC^2 = AB^2 + AC^2 = 3^2 + AC^2 = 9 + AC^2\,.$$

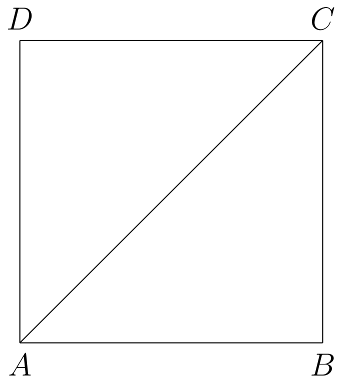
Thus

$$AC^2 = 49 - 9 = 40\,.$$

Therefore $AC = \sqrt{40} = 2\sqrt{10}$.
The area of ABC is $\frac{AB \cdot AC}{2} = \frac{3 \cdot 2\sqrt{10}}{2} = 3\sqrt{10}$.

Exercise 3.6.5 *Find the diagonal of a square with side 1.*

Solution:



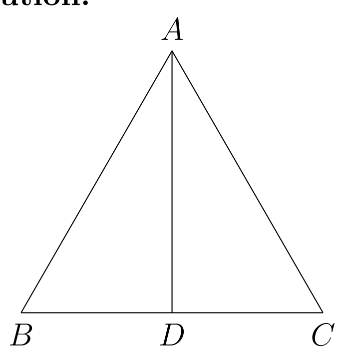
By Pytagora's Theorem

$$AC^2 = AB^2 + BC^2 = 1^2 + 1^2 = 2\,.$$

Therefore $AC = \sqrt{2}$.

Exercise 3.6.6 *Find the area of an equilateral triangle with side* 2.

Solution:



Since ABC is equilateral, we have $BD \equiv toCD$. As $BC = 2$, it follows that $BD = 1$.

By Pytagora's Theorem

$$4 = AB^2 = BD^2 + AD^2 = 1 + AD^2\,.$$

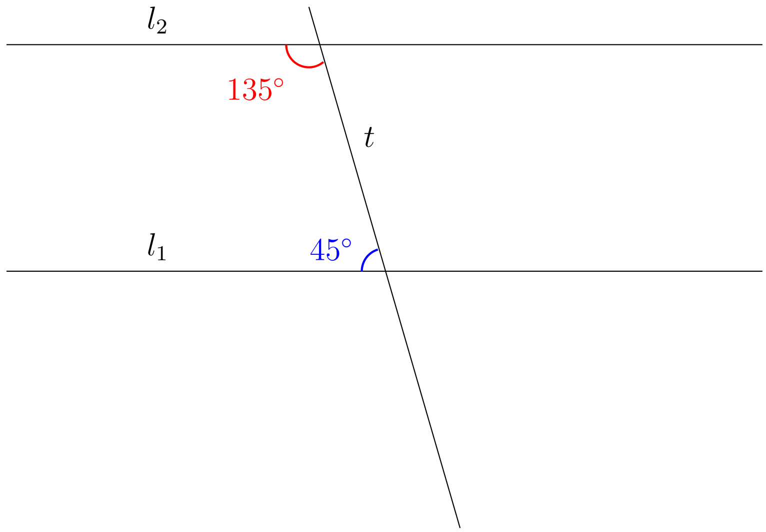
Therefore $AD = \sqrt{3}$.

It follows that the area of ABC is $\frac{2 \cdot \sqrt{3}}{2} = \sqrt{3}$.

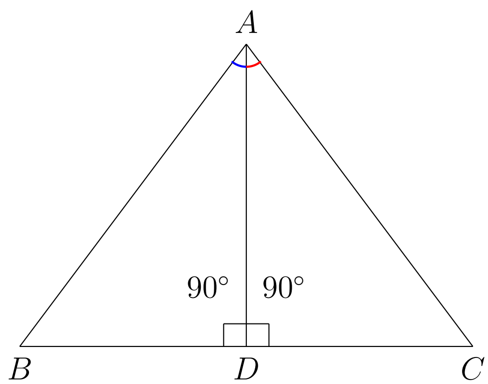
Note: Same way we can prove that the area of a equilateral triangle with side l is $\frac{l^2\sqrt{3}}{4}$.

3.7 Exercises

Exercise 3.7.1 *Prove that the following lines are parallel:*



Exercise 3.7.2 *ABC is a triangle. D is on side BC such that $AD \perp BC$ and $\angle BAD \equiv \angle CAD$.*



Prove that $AB \equiv AC$.

Exercise 3.7.3 *$ABCD$ is a quadrilateral such that $AB \parallel CD$ and $AD \parallel BC$ (i.e. $ABCD$ is a parallelogram). O is the intersection of the diagonals AC and BD. Prove that the triangles AOB and COD are congruent.*

Exercise 3.7.4 *ABC is an isosceles triangle with $\angle A = 30°$. Find the other two angles.*

Exercise 3.7.5 *ABCD are four points on a circle such that AC is a diameter. Find* $\angle BAD + \angle BCD$.

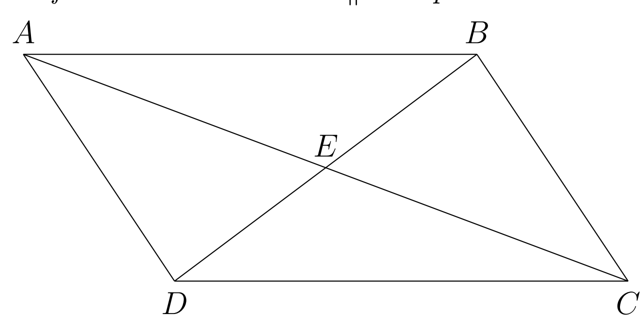
Exercise 3.7.6 *i) ABC is a triangle with* $\angle A = 90°$, $AB = 12, AC = 5$. *Find BC.*

ii) ABC is a triangle with $\angle A = 90°$, $AB = 2, BC = 5$. *Find AC.*

Exercise 3.7.7 *ABC is an isosceles triangle. If one angle is* $70°$ *find the other two angles.*

Exercise 3.7.8 *A triangle ABC has* $\angle A = 90°$, $AB = 1$ *and* $BC = 4$. *Find AC.*

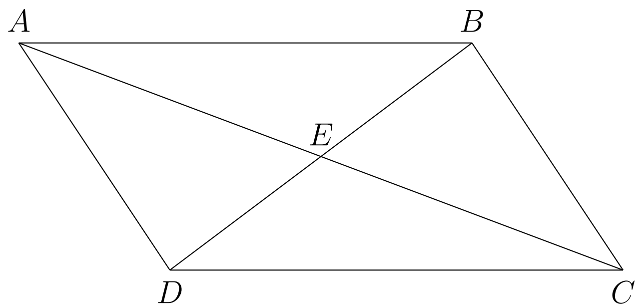
Exercise 3.7.9 *The diagonals AC and BD of a quadrilateral ABCD meet at E. If* $AB \equiv CD$ *and* $AB \parallel CD$ *prove that* $AE \equiv CE$.



Exercise 3.7.10 *ABC is an isosceles triangle. If one angle is* $80°$ *find the other two angles.*

Exercise 3.7.11 *A triangle ABC has* $\angle A = 90°$, $AB = 2$ *and* $BC = 5$. *Find AC.*

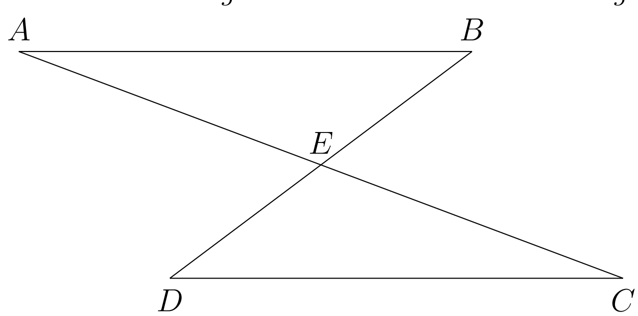
Exercise 3.7.12 *The diagonals AC and BD of a quadrilateral ABCD meet at E. If* $AE \equiv CE$ *and* $BE \equiv DE$ *prove that* $AB \equiv CD$.



Exercise 3.7.13 *A triangle ABC has $\angle B = 30°$ and $\angle C = 50°$. What is the exterior angle A?*

Exercise 3.7.14 *A triangle ABC has $\angle A = 90°$, $AB = 3$ and $AC = 5$. Find BC.*

Exercise 3.7.15 *In the figure below we know that $AB \parallel CD$ and $AB \equiv CD$. Prove that the triangles ABE and CDE are congruent.*



Chapter 4

Introduction to Combinatorics

4.1 Addition and multiplication principle

Addition principle: If two sets have no common elements, then the total number of elements in the two sets is the sum of elements in each set.

The addition principle also tells us that if we want to chose one object from one of two groups, groups which have no common element, the number of possible choices we have is the sum of the number of elements in groups.

Exercise 4.1.1 *There are 15 Pepsi bottles and 18 Coca Cola bottles in a box. How many bottles are in the box?*

Solution: By the addition principle there are $15 + 18 = 33$ bottles in the box.

Exercise 4.1.2 *John sold 12 issues of Sun and 7 issues of Edmonton Journal. How many newspapers did he sell?*

Solution: By the addition principle he sold $12 + 7 = 19$ newspapers.

Exercise 4.1.3 *A coffee shop has 5 brands of tea and 4 types of coffee. In how many ways can we get a tea OR a coffee?*

Solution: By the addition principle there are $5 + 4 = 9$ different drinks we can chose. Thus we can get one tea or one coffee in 9 different ways.

Multiplication principle: If we want to chose one item from a group of M elements and another item from a group of N elements, the total number of two-items choices is $M \cdot N$.

Exercise 4.1.4 *A restaurant shop has 3 choices of soup and 6 main courses. In how many ways can we chose one soup AND one main course?*

Solution: By the multiplication principle there are $3 \cdot 6 = 18$ different ways.

Note: It is easy to see why in these situation we have to multiply the number of choices. In this exercise, we can first pick one of the three soups. If we pick the first soup, we have 6 choices for the main course. If we pick the second soup, we again have 6 choices for the main course. If we pick the third soup, we again have 6 choices for the main course.

Note that choices from different groups have different soups, thus they are different. This means that by the addition principle, the total number of choices is

$$6 + 6 + 6 = 3 \cdot 6 \,.$$

Exercise 4.1.5 *Jake has 3 pairs of jeans and 4 shirts. In how many ways can he dress using one shirt and one pair of jeans?*

Solution: By the multiplication principle there are $3 \cdot 4 = 12$ different ways.

If we have more thant 2 groups, the multiplication principle can be generalized the following way:

The Fundamental Counting Principle: If we have two or more groups of items, and we want to pick exactly one item from each group, the number of ways of choosing the items is obtained by multiplying the number of elements in the groups.

Exercise 4.1.6 *A test consists of* 5 *multiple choice questions, each with* 3 *choices. John is not prepared for the test, and answers the questions at random. In how many different ways can he answer the test?*

Solution: As he chooses exactly one answer from each group of 3 potential answers, by the Fundamental Counting Principle the number of ways is

$$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243 \,.$$

4.1.1 Inclusion Exclusion Principle

In this section we study what happens if we add together two groups of elements with common items.

Inclusion exclusion principle: If we have two groups of items, the total number if items in the two groups is the sum of the items in each group minus the number of common elements.

Note: If the two groups have no common items, then the Inclusion Exclusion principle tells us exactly the same thing as the addition principle.

Exercise 4.1.7 *There are 23 students in a Math class and 25 in English. If there are 5 students taking both classes, how many students are in total in the two classes?*

Solution: By the inclusion exclusion principle there are

$$23 + 25 - 5 = 43\,,$$

students in the two classes.

Note: The reason why the inclusion exclusion principle works is because if we add the number of students in Math and English, we get all the students, but we counted the common students twice : once in math and once in English. Therefore, the sum $23 + 25$ is the number of students plus 5. Subtracting 5 we get the number of students.

Exercise 4.1.8 *How many numbers between* 1 *and* 100 *inclusive are divisible by* 2 *or* 5*?*

Solution: There are 50 even numbers up to 100. There are 20 numbers divisible by 5 up to 100. But since there are common elements (for example 10 appears in both groups), we cannot use the addition principle, we must use inclusion-exclusion.

We now figure out how many common elements are. A number is common if it is divisible by both 2 and 5. Since $gcd(2, 5) = 1$ a number is divisible by 2 and 5 if it is divisible by 10. There are 10 numbers up to 100 which are divisible by 10.

Therefore, by the inclusion-exclusion principle there are

$$50 + 20 - 10 = 60$$

numbers between 1 and 100 inclusive that are divisible by 2 or 5.

Exercise 4.1.9 *There are* 21 *students in Math and* 24 *in English. If there are* 37 *students in total in the two classes, how many are taking both Math and English.*

Solution: Let us denote by x the number of students which take both classes. Then, by the inclusion-exclusion principle we have

$$37 = 21 + 24 - x\,.$$

Solving we get $x = 8$ students are taking both classes.

4.2 Permutations, Arrangements and Combinations

4.2.1 Permutations

A **permutation** is an arrangement of some elements.

In this section we try to answer to the following general question: How many different permutations are for some elements.

Example 4.2.1 *There are 6 permutations of red , blue and green:*

red	*blue*	*green*
red	*green*	*blue*
blue	*red*	*green*
blue	*green*	*red*
green	*blue*	*red*
green	*red*	*blue*

Theorem 4.2.1 *The number of permutations of* n *elements is* $n \cdot (n-1) \cdot (n-2) \cdot ... \cdot 1$.

Proof: We have n choices for the first position. After making this choice, we have $n-1$ elements left, thus we have $n-1$ choices for the second position. Same way we have $n-2$ choices for the second position,.... , and one choice for the last position.

Since we need to make a choice for the first position AND a choice for the second position AND ... AND a choice for the last position, we need to use the Fundamental Counting Principle. By the Fundamental Counting Principle we have

$$n \cdot (n-1) \cdot ... \cdot 1$$

choices. □

Notation: We use the notation $n!$ to denote the product of numbers between 1 and n:

$$n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot n \,.$$

By convention we will use $0! = 1$.

Exercise 4.2.1 *Find* $6!$.

Solution:

$$6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 6 \cdot 20 \cdot 6 = 120 \cdot 6 = 720 \,.$$

Exercise 4.2.2 *In how many ways can we arrange* 5 *books on a shelf?*

Solution: By Theorem 4.2.1, the number of ways is $5! = 120$.

Exercise 4.2.3 *Find* $\frac{10!}{8!}$.

Solution:

$$\frac{10!}{8!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \cdot 9 \cdot 10}{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}} = 90$$

4.2.2 Permutations with selection

Exercise 4.2.4 *A hockey team has* 18 *skaters. The coach needs to chose in order* 3 *players for the shootout. In how many ways can he select them?*

Solution: We have 18 choices for the first shooter. For the second shooter we only have 17 choices left. For the last shooter we have 16 choices left.

By the Fundamental Counting Principle we have

$$18 \cdot 17 \cdot 16 = 4896\,,$$

different ways of choosing in order the three shooter.

Same way as this problem we can prove the following formula:

Theorem 4.2.2 *The number of possible permutations of r elements taken from n is*

$${}_nP_r = \frac{n!}{(n-k)!} = n \cdot (n-1) \cdot ... \cdot (n-r+1)\,.$$

Note: This is the same as choosing in order r elements from n possibilities.

Exercise 4.2.5 *A TV station needs to assign three movies for* $6:00, 8:00$ *and* $10:00$ *time slots. If they have* 5 *movies available, in how many ways can they assign the three time slots.*

Solution: The number of ways is

$${}_5P_3 = \frac{5!}{(5-3)!} = 5 \cdot 4 \cdot 3 = 60\,.$$

4.2.3 Combinations

Combinations count the number of ways of picking r objects from a group of n, **if the order of picking is not important** .

Exercise 4.2.6 *We have four markers:* **black**, *red* , *blue and green. In how many ways can we chose* 2 *colored markers?*

Solution: We can chose 2 markers in 6 different ways:

red	blue
red	green
red	**black**
blue	green
blue	**black**
green	**black**

Theorem 4.2.3 *The number of permutations of ways of choosing r elements from a group of n elements,* **if the order is not important***, is*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

$\binom{n}{k}$ is also called the **binomial coefficient**.

Exercise 4.2.7 *There are 32 students in your class. The class requires team projects, and each team consists of 4 students. In how many ways can you chose three other students for a team project?*

Solution: There are 31 other students in the class. We must chose 3 students from these 31. This can be done in

$$\binom{31}{3} = \frac{31!}{3!28!} = \frac{31 \cdot 30 \cdot 29}{1 \cdot 2 \cdot 3} = 4495$$

ways.

Exercise 4.2.8 *There are 7 singers auditing for a 3 singers group. In how many ways can the 3 singers be selected?*

Solution: The number of ways is

$$\binom{7}{3} = \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35.$$

ways.

4.2.4 Pascal Triangle

We build a triangle the following way: We start with a single row, which we will call row 0, with a 1 in the center.

We then build, step by step row 1, row 2, ... the following way. In each row, a number is obtain by adding the two numbers diagonally above it. If there is no number above it, we consider it to be 0.

We get the following table, which can be continued forever:

Row 0:												1											
Row 1:											1		1										
Row 2:										1		2		1									
Row 3:									1		3		3		1								
Row 4:								1		4		6		4		1							
Row 5:							1		5		10		10		5		1						
Row 6:						1		6		15		20		15		6		1					
Row 7:					1		7		21		35		35		21		7		1				
Row 8:				1		8		28		56		70		56		28		8		1			
Row 9:			1		9		36		84		126		126		84		36		9		1		
Row 10:		1		10		45		120		210		252		210		120		45		10		1	
Row 11:	1		11		55		165		330		462		462		330		165		55		11		1

This triangle is called the **Pascal triangle**.

Important: The nth row in the Pascal triangle is, in order

$$\binom{n}{0} \binom{n}{1} \binom{n}{2} \binom{n}{3} \dots \binom{n}{n}$$

Example 4.2.2 *The sixth row in the Pascal triangle tells us that*

$$\binom{6}{0} = 1\,; \binom{6}{1} = 6\,; \binom{6}{2} = 15\,; \binom{6}{3} = 20\,; \binom{6}{4} = 15\,; \binom{6}{5} = 6\,; \binom{6}{6} = 1\,.$$

The Pascal triangle is a very useful way of calculating many binomial coefficients at once.

4.2.5 Patterns in the Pascal Triangle

The Pascal triangle has the following interesting properties:

Property 1: The Pascal triangle is symmetric. Every row is the same red forward and backward.

Property 2: The sum of elements in row n is 2^n.

For example, the sum of elements in row 5 is

$$1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5 .$$

Property 3: If we pick any row $n \geq 1$, we pick the first number subtract the second and we keep alternating the addition and subtracting, we always get 0.

For example, using the fourth row we get:

$$1 - 4 + 6 - 4 + 1 = 0 .$$

Same way, using 5th row, we get:

$$1 - 5 + 10 - 10 + 5 - 1 = 0 .$$

Property 4: The first and last diagonals consist of all $1'$.

Property 5: The second and second last diagonals consist of the integers in order:

$$1\,2\,3\,4\,5 \ldots .$$

Property 6: The third and third last diagonals consist of the triangular numbers:

$$\begin{aligned} 1 &= 1 \\ 1 + 2 &= 3 \\ 1 + 2 + 3 &= 6 \\ 1 + 2 + 3 + 4 &= 10 \\ 1 + 2 + 3 + 4 + 5 &= 15 \end{aligned}$$

4.2.6 Binomial Theorem

The binomial coefficients are also important because of the following theorem:

Theorem 4.2.4 *(***Binomial Theorem***) Let n be a positive integer. Then the formula for $(a+b)^n$ is*

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a b^{n-1} + b^n .$$

The coefficients are exactly the n th row of the Pascal triangle.

For example, the second row of the Pascal triangle is 121. The Binomial Theorem tell us that

$$(a+b)^2 = a^2 + 2ab + b^2 \,.$$

Same way, the 6th row tells us that

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \,.$$

4.3 Pigeon Hole Principle

The **Pigeon Hole Principle** says that if we split some objects into groups, and we have more objects than groups, then at least one group must contain multiple objects.

It is often stated in the following form:

Pigeon Hole Principle: If some pigeons go inside some pigeonholes, and there are more pigeons than holes, then at least one hole has more than one pigeons.

Exercise 4.3.1 *If we have three gloves, then there must be (at least) two left gloves or two right gloves in our group.*

This is a simple application of the pigeon hole principle. Our objects (pigeons) are the gloves and our groups (pigeonholes) are the type of gloves. As we have 3 objects and only 2 groups, there must be more than one object in a group.

Exercise 4.3.2 *If there are 13 or more people in a room, there are two people whose birthday is in the same month.*

Our elements are the people, and the groups are given by month they are born.

Exercise 4.3.3 *If we pick six numbers between 1 and 10, two of these numbers must add to 11.*

Solution: Our objects are the six numbers we pick. Now we group them such that the sum is 11:

$$\text{group i} : \{1, 10\}$$
$$\text{group ii} : \{2, 9\}$$
$$\text{group iii} : \{3, 8\}$$
$$\text{group iv} : \{4, 7\}$$
$$\text{group v} : \{5, 6\}$$

As we have 6 objects and only 5 groups, two of our objects (numbers) must be in the same group. Those two numbers add to 11.

Exercise 4.3.4 *If we pick six numbers between 1 and 10, there must be two consecutive numbers in our group.*

Solution: Our objects are the six numbers we pick. Now we group them in such way that we get 5 groups of consecutive numbers:

$$\text{group i} : \{1, 2\}$$
$$\text{group ii} : \{3, 4\}$$
$$\text{group iii} : \{5, 6\}$$
$$\text{group iv} : \{7, 8\}$$
$$\text{group v} : \{9, 10\}$$

As we have 6 objects and only 5 groups, two of our objects (numbers) must be in the same group. Those two numbers are consecutive.

4.4 Exercises

Exercise 4.4.1 *There are 4 brands of coffee and 3 brands of tea. In how many ways can we chose one coffee and one tea?*

Exercise 4.4.2 *In how many ways cane we arrange the letters* A, B, C, D, E *is a row?*

Exercise 4.4.3 *There are 7 horses in a race. In how many ways can the first three horses come in?*

Exercise 4.4.4 *In how many ways cane we chose 3 different letters in the English alphabet?*

Exercise 4.4.5 *A school offers 6 MATH classes and 9 ENG classes. In how many ways can a student choose one class?*

Exercise 4.4.6 *There are 10 books in a store. In how many ways can someone buy three books from the store?*

Exercise 4.4.7 *A school offers 7 MATH classes and 8 ENG classes. In how many ways can a student take a Math and a ENG class?*

Exercise 4.4.8 *There are 10 problems at the end of a section in a book. A student wants to solve 3 of these problems. In how many ways can he chose 3 problems from the list of 10?*

Exercise 4.4.9 *A pet store has 5 dogs and 6 cats. If Jon wants to buy a dog and a cat, in how many ways can he do this?*

Exercise 4.4.10 *A teacher wishes to assign each of* 5 *students a different book to read. If the teacher has* 9 *available books, in how many ways can this be done?*

Chapter 5

Solutions

5.1 Section 1.1

5.2 Section 1.2

5.3 Section 1.3

5.4 Section 2.1

5.5 Section 2.2

Exercise 2.2.6

(a) Since $(-3)^3 = -27$ we have $\sqrt[3]{-27} = -3$. Therefore

$$\sqrt[3]{-27} + \sqrt{9} = -3 + 3 = 0\,.$$

(b)

$$\begin{aligned} 2\sqrt{18} + 3\sqrt{27} - 5\sqrt{3} &= 2\sqrt{9 \cdot 2} + 3\sqrt{9 \cdot 3} - 5\sqrt{3} \\ &= 6\sqrt{2} + 9\sqrt{3} - 5\sqrt{3} = 6\sqrt{2} + 4\sqrt{3} \end{aligned}$$

Exercise 2.2.7

$$\begin{aligned} 9^{\frac{1}{n}} &= 3^{\frac{1}{6}} \\ (3^2)^{\frac{1}{n}} &= 3^{\frac{1}{6}} \\ 3^{\frac{2}{n}} &= 3^{\frac{1}{6}} \end{aligned}$$

Therefore

$$\frac{2}{n} = \frac{1}{6}\,.$$

Cross-multiplication gives us

$$2 \cdot 6 = 1 \cdot n\,.$$

Thus

$$n = 12\,.$$

5.6 Section 2.3

Exercise 2.3.8

(a)

$$\begin{aligned} 3(3x+1) &< 4(5x+2) \\ 9x+3 &< 20x+8 \\ 9x-20x &< 8-3 \\ -11x &< 5 \\ x &> -\frac{5}{11} \end{aligned}$$

Answer: $(-\frac{5}{11}, \infty)$.

(b)

$$\begin{aligned} 2x+1 &\leq 3x+2 \\ 2x-3x &\leq 2-1 \\ -x &\leq 1 \\ x &\geq -1 \end{aligned}$$

$[-1, \infty)$.

$$\begin{aligned} 3x + 2 &\le 2x + 7 \\ 3x - 2x &\le 7 - 2 \\ x &\le 5 \end{aligned}$$

$(-\infty, 5]$.

Answer $(-\infty, 5] \cap [-1, \infty) = [-1, 5]$.

5.7 Section 2.4

Exercise 2.4.6

(a) Doubling the first equation we get

$$\begin{cases} 6x + 4y = 8 \\ 6x + 4y = 4 \end{cases}.$$

Subtracting, we get

$$0 = 4\,.$$

There is no solution.

(b)

$$\begin{cases} 2x + 3y + 7z = 5 \\ 5x + 7y + 2z = 8 \end{cases}.$$

Multiplying the first equation by 5 and second by 2 we get

$$\begin{cases} 10x + 15y + 35z = 25 \\ 10x + 14y + 4z = 16 \end{cases}.$$

Subtracting we get

$$y + 31z = 9\,.$$

We now look at first and third.

$$\begin{cases} 2x + 3y + 7z = 5 \\ 3x + 4y - 4z = 2 \end{cases}.$$

Multiplying the first equation by 3 and second by 2 we get

$$\begin{cases} 6x + 9y + 21z = 15 \\ 6x + 8y - 8z = 4 \end{cases} .$$

Subtracting we get

$$y + 29z = 11 .$$

Therefore we get the smaller system

$$\begin{cases} y + 31z = 9 \\ y + 29z = 11 \end{cases} .$$

Subtracting we get $2z = -2$ hence $z = -1$.

The first equation tells us

$$y - 31 = 9 .$$

therefore $y = 40$.

Then

$$2x + 3 \cdot 40 + 7(-1) = 5 \Rightarrow 2x = 12 - 120 \Rightarrow x = -54 .$$

The answer is

$$\begin{cases} x &= -54 \\ y &= 40 \\ z &= -1 \end{cases} .$$

5.8 Section 2.5

Exercise 2.5.2

$$x^2 + 3x + 1 = x^2 + 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2 + 1 = (x + \frac{3}{2})^2 - \frac{9}{4} + 1 = (x + \frac{3}{2})^2 - \frac{5}{4} .$$

5.9 Section 2.6

Exercise 2.6.6

(a)

$$\Delta = 4 - 4 \cdot 3 \cdot 0 = 4\,.$$

There are two solutions

$$x_{1,2} = \frac{-2 \pm \sqrt{4}}{2 \cdot 3} = \frac{-2 \pm 2}{6}\,.$$

The solutions are $x = 0$ an $x = -\frac{2}{3}$.

(b)

$$\Delta = 9 - 4 \cdot 7 < 0\,.$$

There is no solution.

5.10 Section 2.7

5.11 Section 2.8

5.12 Section 2.9