Trent University, Summer 2016

MATH 1001H Test

Tuesday, 31 May, 2016

Time: 60 minutes

Name: Solutions

Student Number: 0000000

Question	**Mark**	
1	____	
2	____	
3	____	
Total	____	/30

Instructions

- *Show all your work.* Legibly, if possible!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Solve for x as best you can in any *four* (4) of **a**–**f**. *[12 = 4 × 3 each]*

a. $\log_2(x-2) = 2$ **b.** $x^2 + 2x + 1 = 0$ **c.** $|x-3| = 2$

d. $\sin^2(x) = \dfrac{3}{4}$ **e.** $10^{2x+1} = 0.001$ **f.** $\tan^{-1}(x) = -45°$

SOLUTIONS. **a.** $\log_2(x-2) = 2 \Rightarrow x - 2 = 2^{\log_2(x-2)} = 2^2 = 4 \Rightarrow x = 4 + 2 = 6$. □

b. If you observe that $x^2 + 2x + 1 = (x+1)^2$, this over with very quickly: $(x+1)^2 = 0 \Leftrightarrow x = -1$. Otherwise, you can work a little harder and apply the quadratic formula. □

c. $|x-3| = 2 \Rightarrow x - 3 = \pm 2 \Rightarrow x = 3 \pm 2 \Rightarrow x = 3 + 2 = 5$ or $x = 3 - 2 = 1$. □

d. $\sin^2(x) = \dfrac{3}{4} \Rightarrow \sin(x) = \pm\sqrt{\dfrac{3}{4}} = \pm\dfrac{\sqrt{3}}{2}$. Recalling that $\sin(60°) = \sin(120°) = \frac{\sqrt{3}}{2}$, and that $\sin(t \pm n \cdot 360°) = \sin(t)$ for every real number t and every integer n, we get that $x = 60° \pm n \cdot 360°$ and $x = 120° \pm n \cdot 360°$ satisfy the equation for every integer n. Similarly, recalling that $\sin(-60°) = \sin(-120°) = -\frac{\sqrt{3}}{2}$, and that $\sin(t \pm n \cdot 360°) = \sin(t)$ for every real number t and every integer n, we get that $x = -60° \pm n \cdot 360°$ and $x = -120° \pm n \cdot 360°$ also satisfy the equation for every integer n.

Thus $x = \pm 60° \pm n \cdot 360°$ for any integer n and $x = \pm 120° \pm n \cdot 360°$ for any integer n are all the possible solutions of the equation $\sin^2(x) = \frac{3}{4}$. Whew! □

e. $10^{2x+1} = 0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3} \Rightarrow 2x + 1 = -3 \Rightarrow 2x = -3 - 1 = -4$
$\Rightarrow x = -\frac{4}{2} = -2$. □

f. $\tan^{-1}(x) = -45°$ exactly when $x = \tan(-45°) = \dfrac{\sin(-45°)}{\cos(-45°)} = \dfrac{-1/\sqrt{2}}{1/\sqrt{2}} = -1$. ■

2. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. Suppose that $\cos(\alpha) = \dfrac{12}{13}$. Compute each of:

i. $\sin(\alpha)$ *[1]* *ii.* $\tan(\alpha)$ *[1]* *iii.* $\sec(\alpha)$ *[1]* *iv.* $\sin(2\alpha)$ *[1]* *v.* $\cos(2\alpha)$ *[1]*

b. Sketch the graphs of: *i.* $y = e^x$ *[1]* *ii.* $y = e^{-x}$ *[1]* *iii.* $y = \dfrac{e^x + e^{-x}}{2}$ *[1.5]*

iv. At what point(s), if any, do the graphs of these functions intersect? *[1.5]*

c. Let $f(x) = 2x + |x|$. *i.* Sketch the graph of $y = f(x)$. *[2]*

ii. Find the inverse function, $f^{-1}(x)$, of $f(x)$. *[3]*

SOLUTIONS. **a.** *i.* $\sin(\alpha) = \sqrt{1-\cos^2(\alpha)} = \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$. □

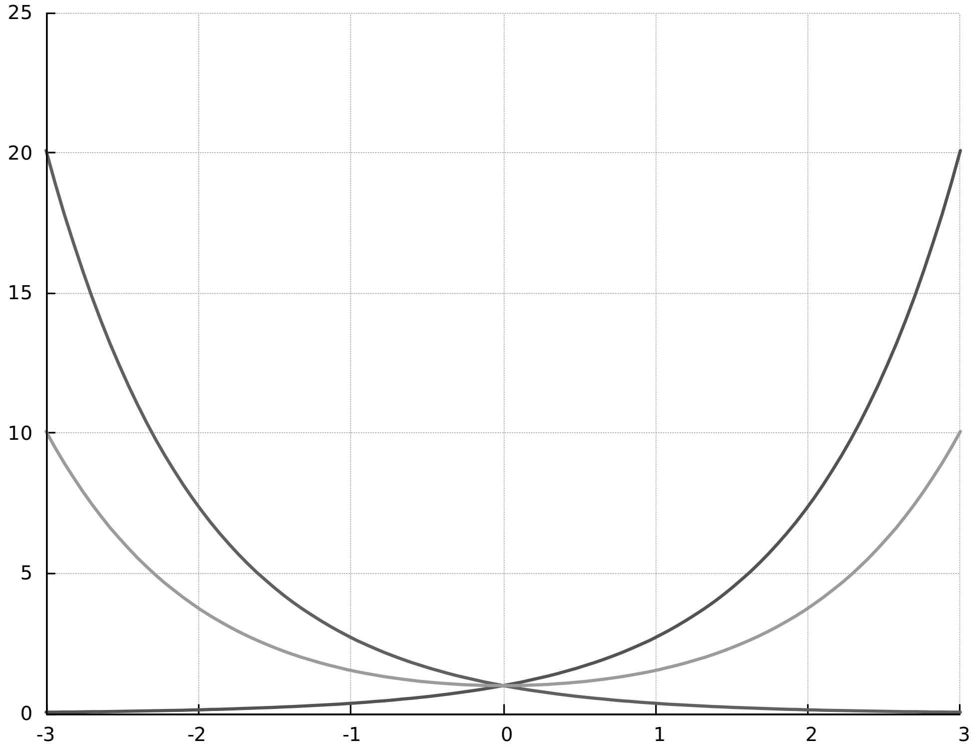
ii. $\tan(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)} = \dfrac{5/13}{12/13} = \dfrac{5}{12}$. □

iii. $\sec(\alpha) = \dfrac{1}{\cos(\alpha)} = \dfrac{1}{12/13} = \dfrac{13}{12}$. □

iv. $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) = 2 \cdot \dfrac{5}{13} \cdot \dfrac{12}{13} = \dfrac{120}{169}$. □

v. $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = \left(\dfrac{12}{13}\right)^2 - \left(\dfrac{5}{13}\right)^2 = \dfrac{144}{169} - \dfrac{25}{169} = \dfrac{119}{169}$. □

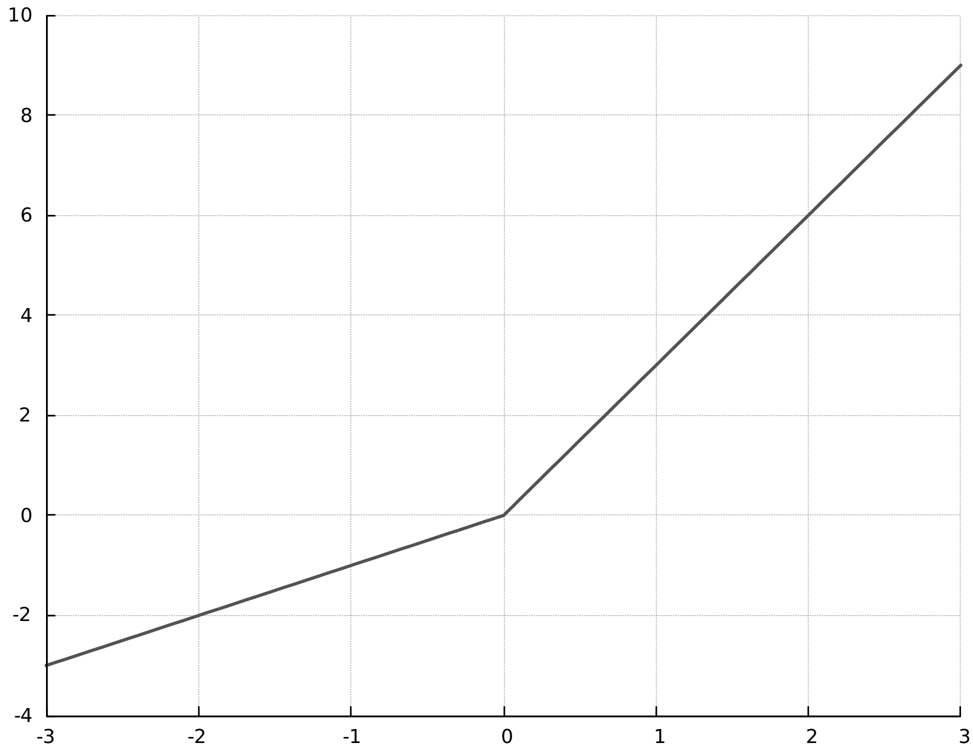
b. *i–iii.* Here is a graph of all three functions on a single set of axes. The increasing one is $y = e^x$, the decreasing one is $y = e^{-x}$, and the one with a minimum at $x = 0$ is $y = \dfrac{e^x + e^{-x}}{2} = \cosh(x)$.



□

iv. Since $e^0 = 1$, we have $e^0 = e^{-0} = 1 = \frac{1+1}{2} = \frac{e^0+e^{-0}}{2}$, so all three functions pass through the point $(0, 1)$. It's pretty easy to see from the graph that this is the only point of intersection any two of them have. □

c. *i.* Here is a graph of $y = 2x + |x|$. Note that when $x \geq 0$, $2x + |x| = 2x + x = 3x$, and when $x \leq 0$, $2x + |x| = 2x + (-x) = x$.



□

ii. As noted above, $f(x) = 2x + |x| = \begin{cases} 3x & x \geq 0 \\ x & x \leq 0 \end{cases}$. Since the inverse function of $g(x) = x$ is just $g^{-1}(x) = g(x) = x$, and the inverse function of $h(x) = 3x$ is $h^{-1} = \frac{x}{3}$, it follows that $f^{-1}(x) = \begin{cases} \frac{x}{3} & x \geq 0 \\ x & x \leq 0 \end{cases}$. ■

3. Do *all three* of **a**–**c**. *[8]*

a. Find the equation of the line passing through both $(1,2)$ and $(4,5)$ and sketch the line. *[2]*

b. Find the location of the tip of the parabola given by $y = x^2 - 4x + 5$ and sketch the parabola. *[4]*

c. Find all the points of intersection, if any, of the line in **a** and the parabola in **b**. *[2]*

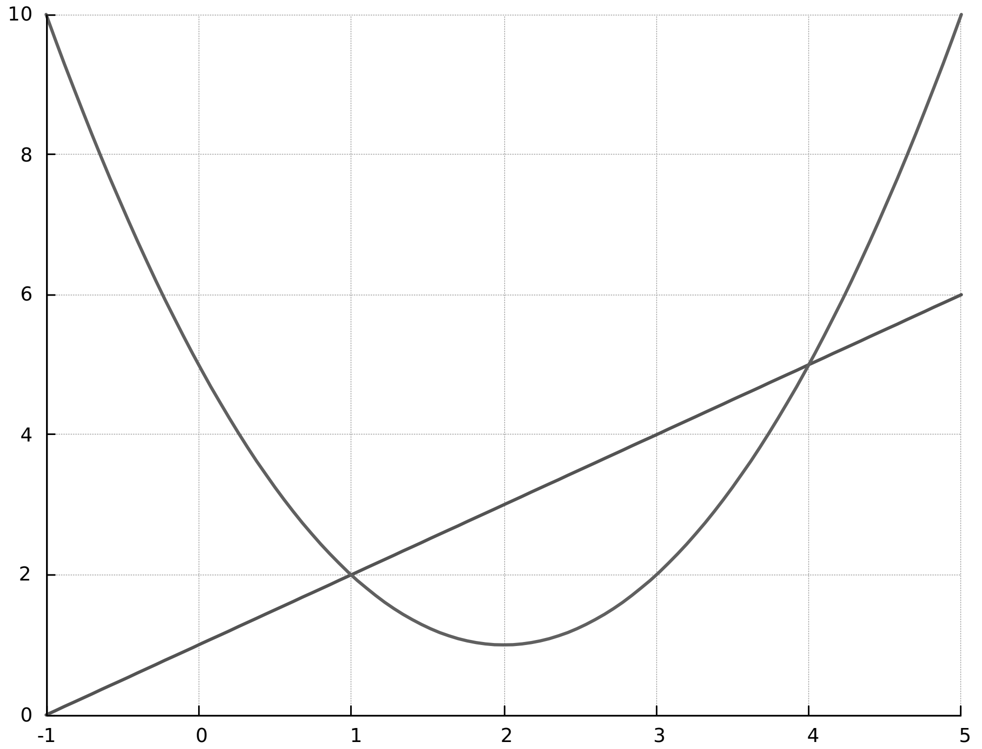
SOLUTIONS. **a.** The line through $(1,2)$ and $(4,5)$ has slope $m = \frac{5-2}{4-1} = \frac{3}{3} = 1$. We compute the y-intercept, b, by plugging one of the points into $y = 1x + b$ and solving for b: $2 = 1 \cdot 1 + b \Rightarrow b = 2 - 1 = 1$. Thus the equation of the line is $y = x + 1$. The graph of the line is given below together with the graph of the parabola from **b**. □

b. To find the tip of the parabola, we complete the square in the expression defining the parabola:

$$\left[x^2 - 4x\right] + 5 = \left[\left(x + \frac{-4}{2}\right)^2 - \left(\frac{-4}{2}\right)^2\right] + 5 = \left[(x-2)^2 - 4\right] + 5 = (x-2)^2 + 1$$

It follows that the tip of the parabola is at $x = 2$ and $y = 1$, *i.e.* at the point $(2,1)$. □

Here is a graph of the line $y = x + 1$ from **a** and the parabola $y = x^2 - 4x + 5$ from **b**:



c. It's pretty obvious from the graph that the line and the parabola intersect at the points $(1,2)$ and $(4,5)$. Algebraically, observe that at any point of intersection (x,y), we must have $x^2 - 4x + 5 = y = x + 1$, so $0 = \left(x^2 - 4x + 5\right) - (x + 1) = x^2 - 5x + 4$. we can solve this equation using the quadratic formula:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1} = \frac{5 \pm \sqrt{25 - 16}}{2} = \frac{5 \pm \sqrt{9}}{2} = \frac{5 \pm 3}{2}$$

Thus the line and the parabola intersect when $x = \frac{5+3}{2} = \frac{8}{2} = 4$ and when $x = \frac{5-3}{2} - \frac{2}{2} = 1$. Plugging these values for x into the equations of the line (or the parabola, if you want to work harder :-) gives us the corresponding values for y of $4 + 1 = 5$ and $1 + 1 = 2$, respectively, giving the points of intersection $(1, 2)$ and $(4, 5)$. ■

[Total = 30]