Mathematics 1001H – Precalculus Mathematics
TRENT UNIVERSITY, Summer 2016

Solutions to Assignment #3
Some Significant Sequence Sums

In all that follows, n is some arbitrary positive integer.

1. Explain why $1+2+3+\cdots+(n-1)+n = \dfrac{n(n+1)}{2}$. *[2]*

NOTE: We did this one in passing in class one day, if memory serves ...

SOLUTION. We did this in class:

$$\begin{aligned}
2\cdot(1+2+\cdots+(n-1)+n) &= (n+(n-1)+\cdots+2+1)+(1+2+\cdots+(n-1)+n) \\
&= [n+1]+[(n-1)+2]+\cdots+[2+(n-1)]+[1+n] \\
&= [n+1]+[n+1]+\cdots+[n+1]+[n+1] = n(n+1)
\end{aligned}$$

so $1+2+\cdots+(n-1)+n = \dfrac{n(n+1)}{2}$. $\square$

2. Suppose a and d are numbers. Find a formula (in terms of a, d, and n) for the sum $a+(a+d)+(a+2d)+\cdots+(a+nd)$. *[2]*

NOTE: The sequence of numbers a, $a+d$, $a+2d$, ... , is the *arithmetic sequence* with first term a and *common difference* d. Their sum is the corresponding *arithmetic series*.

SOLUTION. Here goes; just regroup and apply the formula from problem **1**:

$$\begin{aligned}
a+(a+d)+(a+2d)+\cdots+(a+nd) &= (n+1)a+(1+2+\cdots+n)d \\
&= (n+1)a+\frac{n(n+1)}{2}\cdot d \quad \square
\end{aligned}$$

3. Suppose a and $r>0$ are numbers. Find a formula (in terms of a, r, and n) for the sum $a+ar+ar^2+ar^3+\cdots+ar^n$. *[3]*

Hint: Try multiplying the sum by $1-r$ and see what happens.

NOTE: The sequence of numbers a, ar, ar^2, ... , is the *geometric sequence* with first term a and *common ratio* r. Their sum is the corresponding *geometric series*.

SOLUTION. Following the hint:

$$\begin{aligned}
\left(a+ar+ar^2+ar^3+\cdots+ar^n\right)(1-r) &= \left(a+ar+ar^2+ar^3+\cdots+ar^n\right) \\
&\quad - \left(a+ar+ar^2+ar^3+\cdots+ar^n\right)r \\
&= a+ar+ar^2+ar^3+\cdots+ar^n \\
&\quad - ar-ar^2-ar^3-\cdots-ar^n-ar^{n+1} \\
&= a-ar^{n+1} = a\left(1-r^{n+1}\right)
\end{aligned}$$

Dividing both sides by $1-r$ gives us $a + ar + ar^2 + ar^3 + \cdots + ar^n = a \cdot \dfrac{1-r^{n+1}}{1-r}$. $\square$

4. Suppose a, d, and $r > 0$ are numbers. Let $a_0 = 0$ and, for each $k \geq 0$, let $a_{k+1} = a_k r + d$. (Thus $a_1 = ar + d$, $a_2 = (ar+d)r + d$, $a_3 = [(ar+d)r+d]\,r + d$, and so on.) Find a formula (in terms of a, d, r, and n) for the sum $a_0 + a_1 + a_2 + \cdots + a_n$. *[3]*

Hint: Use algebra to take it all apart and reassemble it in another way. Keep your answers to the previous questions in mind, too.

NOTE: There's a name for this, but – darn it! – I can't remember it ...

SOLUTION. Following the hint, and using the formula obtained in the solution to **3**:

$$\begin{aligned}
a_0 + a_1 + a_2 + \cdots + a_n &= a + [ar+d] + ([ar+d]r + d) + [([ar+d]r+d)\,r + d] + \cdots \\
&= a + [ar+d] + \left[ar^2 + dr + d\right] + \left[ar^3 + dr^2 + dr + d\right] + \\
&\quad \cdots + \left[ar^n + dr^{n-1} + dr^{n-2} + \cdots + dr + d\right] \\
&= a\left[1 + r + r^2 + r^3 + \cdots + r^n\right] + \\
&\quad d\left[1 + (1+r) + \left(1 + r + r^2\right) + \cdots + \left(1 + r + \cdots + r^{n-1}\right)\right] \\
&= a \cdot \frac{1-r^{n+1}}{1-r} + d\left[\frac{1-r}{1-r} + \frac{1-r^2}{1-r} + \cdots + \frac{1-r^n}{1-r}\right] \\
&= a \cdot \frac{1-r^{n+1}}{1-r} + \frac{d}{1-r}\left[(1 + \cdots + 1) - \left(r + r^2 + \cdots + r^n\right)\right] \\
&= a \cdot \frac{1-r^{n+1}}{1-r} + \frac{d}{1-r}\left[n - r\left(1 + r + \cdots + r^{n-1}\right)\right] \\
&= a \cdot \frac{1-r^{n+1}}{1-r} + \frac{nd}{1-r} - \frac{dr}{1-r} \cdot \frac{1-r^n}{1-r}
\end{aligned}$$

One could rearrange this in various ways, but it probably won't get much neater. $\blacksquare$