Mathematics 1001H – Precalculus Mathematics
TRENT UNIVERSITY, Summer 2016
Solutions to Assignment #2

1. Find three different functions which have domain $\mathbb{R}$ and which are their own inverses (*i.e.* $(f \circ f)(x) = f(f(x)) = x$ for all $x \in \mathbb{R}$). *[3]*

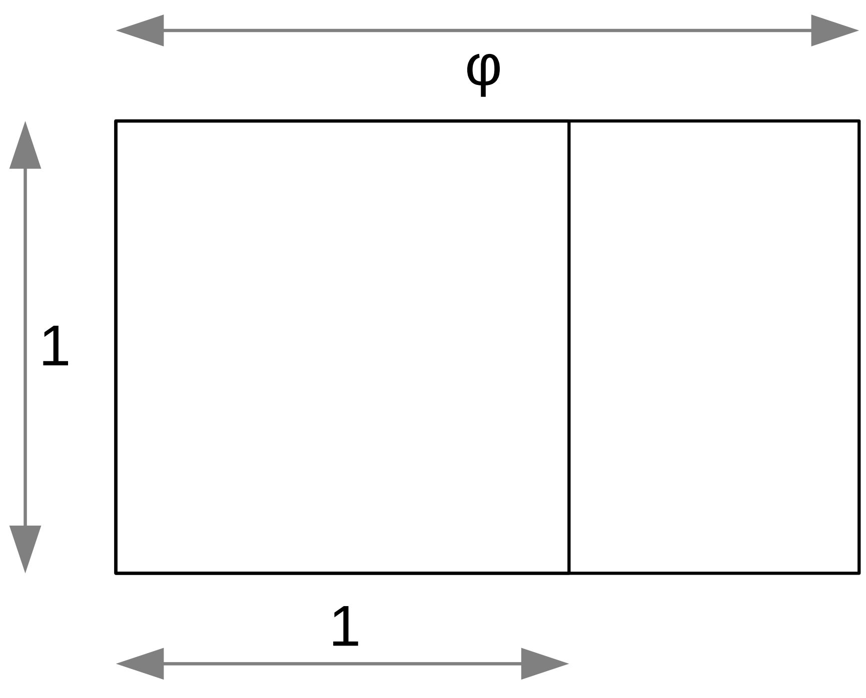
SOLUTION. $f(x) = x$ is one such function, since it is defined for all $x \in \mathbb{R}$ and $f(f(x)) = f(x) = x$ for all $x \in \mathbb{R}$, too.

$f(x) = -x$ is another, since since it is also defined for all $x \in \mathbb{R}$ and we always have $f(f(x)) = f(-x) = -(-x) = x$.

$f(x) = \dfrac{1}{x}$ almost works since $f(f(x)) = f\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} = x$ whenever $x \neq 0$, but it doesn't have domain $\mathbb{R}$ because it isn't defined at $x = 0$. We can fix this by arbitrarily giving it a suitable value at $x = 0$: let $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. This now has domain $\mathbb{R}$. We already checked that $f(f(x)) = x$ for $x \neq 0$, and when $x = 0$, we get $f(f(0)) = f(0) = 0$, so $f(f(x)) = x$ for all x.

That makes three functions with domain $\mathbb{R}$ that are their own inverses. Can you find any more? $\square$

2. The "golden ratio", usually denoted by the lowercase Greek letter phi (which looks like ϕ or φ), is the real number such that if you cut a 1 square from the end of a $1 \times \varphi$ rectangle, the rectangle left over has the same ratio of long side to short side (namely $\frac{\varphi}{1} = \varphi$) as the original rectangle. Use this fact to solve for φ. *[4]*



NOTE. In classical Greece, these proportions for a rectangle were considered to be the most pleasing possible. The Parthenon in Athens, for example, makes repeated use of such proportions.

SOLUTION. We are given that $\varphi = \dfrac{\varphi}{1} = \dfrac{1}{\varphi - 1}$. It follows that $\varphi(\varphi-1) = 1$, so $\varphi^2 - \varphi = 1$, and so $\varphi^2 - \varphi - 1 = 0$. Applying the quadratic formula tells us that:

$$\varphi = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Since $2^2 = 4 < 5$, it follows that $1 < 2 < \sqrt{5}$, so $\dfrac{1-\sqrt{5}}{2} < 0 < 1$. As φ must be greater than 1 for the setup to make sense, it must be the case that $\varphi = \dfrac{1+\sqrt{5}}{2}$. $\square$

3. Find a function $g(x)$ with domain $(0, \infty)$ such that all of

i. $g(64) = 6$,

ii. $g(x)$ is continuous on its domain, and

iii. $g(ab) = g(a) + g(b)$ for all a and b in $(0, \infty)$

are true. Is there more than one such function? *[3]*

SOLUTION. Note that $64 = 2^6$ and we are supposed to have that $g(64) = 6$. Along with the clues that $g(x)$ is supposed to have domain $(0, \infty)$, be continuous, and satisfy $g(ab) = g(a) + g(b)$ for all a and b in $(0, \infty)$ – which three facts are true of all logarithmic functions – this tells us that $g(x) = \log_2(x)$ works. (Note that $\log_2(64) = \log_2\left(2^6\right) = 6$ by the definition of $\log_2(x)$.)

Is there another such function? There isn't. Proving that in full needs some concepts and facts from calculus, especially the notion of limits. (Mind you, you need limits to properly define "continuous" anyway.) Here's a sketch of how you might go about the job:

First, note that $6 = g(64) = g(64 \cdot 1) = g(64) + g(1) = 6 + g(1)$, which means that $g(1) = 6 - 6 = 0$.

Second, we also have:

$$g(64) = g\left(2^6\right) = g(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = g(2) + g(2) + g(2) + g(2) + g(2) + g(2) = 6g(2)$$

Since $6g(2) = g(64) = 6$, we have to have $g(2) = 1$. It's pretty obvious that it follows that $g(2^n) = g(2) + \cdots + g(2) = n$ for every positive integer n.

Third, for any integer $k > 0$, $2 = 2^1 = 2^{k/k} = \left(2^{1/k}\right)^k = 2^{1/k} \cdot \ldots \cdot 2^{1/k}$, so:

$$1 = g(2) = g\left(2^{1/k} \cdot \ldots \cdot 2^{1/k}\right) = g\left(2^{1/k}\right) + \cdots + g\left(2^{1/k}\right) = kg\left(2^{1/k}\right)$$

It follows that $g\left(2^{1/k}\right) = \frac{1}{k}$.

Fourth, combining the facts above, it's pretty easy to see that we must have that $g\left(2^{n/k}\right) = \frac{n}{k}$ for every possible fraction $\frac{n}{k} > 0$. That is, $g\left(2^q\right) = q = \log_2\left(2^q\right)$ for every positive rational number q.

Fifth, continuity (and limits :-) guarantee that if two functions agree on all the numbers of the form 2^q, with $q > 0$, then they must agree on all the positive reals, basically because you can approximate any real number as closely as you like by such a number. (That's where the hardest technical stuff in the argument is buried.) Thus $g(x) = \log_2(x)$ for all $x \in (0, \infty)$.

Whew! $\blacksquare$