Mathematics 1001H – Precalculus Mathematics
TRENT UNIVERSITY, Summer 2016
Solutions to the Quizzes

Quiz #1. Thursday, 12 May, 2016. *[10 minutes]*

1. Compute $\frac{1}{6} + \frac{3}{10}$ and write the answer as a single fraction in lowest terms. *[2]*
2. Find the interval that includes exactly those real numbers x satisfying $|x-3| \leq 2$. *[3]*

SOLUTIONS. 1. The denominators, $6 = 2 \cdot 3$ and $10 = 2 \cdot 5$, have a (least) common multiple of $30 = 5 \cdot 6 = 3 \cdot 10$. We'll write the fractions using 30 as a common denominator, and then find the sum and simplify it:

$$\frac{1}{6} + \frac{3}{10} = \frac{5 \cdot 1}{5 \cdot 6} + \frac{3 \cdot 3}{3 \cdot 10} = \frac{5}{30} + \frac{9}{30} = \frac{5+9}{30} = \frac{14}{30} = \frac{2 \cdot 7}{2 \cdot 15} = \frac{7}{15}$$

For the last couple of steps, note that $14 = 2 \cdot 7$ and $30 = 2 \cdot 3 \cdot 5 = 2 \cdot 15$ have 2 as their only common integer factor other than 1. □

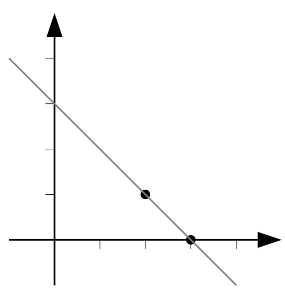
2. The idea here is to try to "solve" the given inequality for x: $|x-3| \leq 2$ means that $-2 \leq x - 3 \leq 2$, which is equivalent to $-2 + 3 \leq x - 3 + 3 \leq 2 + 3$, which is $1 \leq x \leq 5$. The collection of all real numbers x satisfying $1 \leq x \leq 5$ is exactly the interval $[1, 5]$. Note that the endpoints are included. ■

Quiz #2. Tuesday, 17 May, 2016. *[10 minutes]*

Consider the line that passes through the point $(2, 1)$ and has x-intercept 3.

1. Sketch the given line. *[1]*
2. Find the equation of the given line. *[2]*
3. Find the point where the given line intersects the line $y = x + 1$. *[2]*

SOLUTIONS. 1. Here is a crude sketch of the given line:



□

2. Since the points $(2, 1)$ and $(3, 0)$ [the x-intercept] are on the line it must have slope:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{0-1}{3-2} = \frac{-1}{1} = -1$$

Thus the line has an equation of the form $y = mx + b = -x + b$; to determine the y-intercept b, we plug in the coordinates $(x, y) = (3, 0)$ of one of the two given points and solve for b:

$$y = -x + b \Rightarrow 0 = -3 + b \Rightarrow b = 3$$

It folllows that the equation of the given line is $y = -x + 3$. □

3. We know from the answer to question 2 above that the equation of the given line is $y = -x + 3$. To find the point of intersection of the given line and the line given by $y = x + 1$, we set the two equations equal to each other and solve for x:

$$-x + 3 = y = x + 1 \Rightarrow 3 - 1 = x + x \Rightarrow 2x = 2 \Rightarrow x = 1$$

We plug this x value into the equation of either line to get the corresponding y value: $y = -x + 3 = -1 + 3 = 2$. [Check if it's on the other line too: $y = 1 + 1 = 2$. It is!] Thus the point of intersection of the two lines is $(x, y) = (1, 2)$. ■

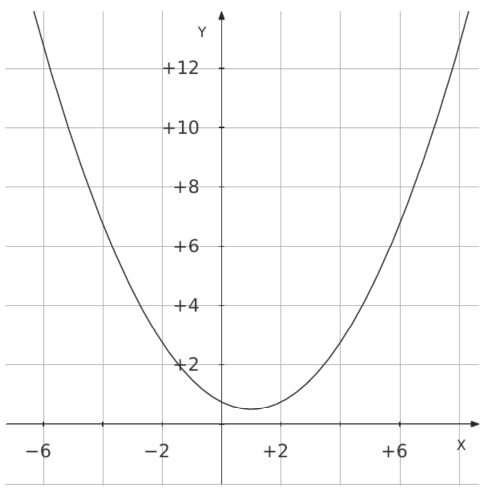
Quiz #3. Thursday, 19 May, 2016. *[10 minutes]*

1. What is the equation of the parabola obtained by shifting the graph of $y = x^2$ 2 units left and 3 units down? *[1.5]*
2. Where is the tip of the parabola given by $y = x^2 - 2x + 3$? *[1.5]*
3. Sketch the parabola given by $y = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{3}{4}$. *[2]*
   [*Hint:* Use your work in answering question 2 ...]

SOLUTIONS. 1. To shift the graph 2 units left, we replace x by $x - (-2) = x + 2$; to shift the graph 3 units down, we simply subract 3. This means that the equation of the shifted parabola is $y = (x + 2)^2 - 3$. (If you expand that, it works out to $y = x^2 + 4x + 1$.) □

2. To locate the tip, we "complete the square" in the expression $x^2 - 2x + 3$. Half of -2 is -1, and $(x - 1)^2 = x^2 - 2x + 1$, so $x^2 - 2x = (x - 1)^2 - 1$. It follows that $y = x^2 - 2x + 3 = (x - 1)^2 - 1 + 3 = (x - 1)^2 + 2$, which tells us that the tip of the parabola is at $x = 1$ (*i.e.* when $x - 1 = 0$) and $y = 2$. That is, the tip of the parabola is at the point $(1, 2)$. □

3. Note that $y = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{3}{4} = \frac{1}{4}\left(x^2 - 2x + 3\right)$, so the graph of this parabola is just the graph of the parabola $y = x^2 - 2x + 3$ stretched by a factor of $\frac{1}{4}$. In particular, the answer to question 2 tells us that the tip is at $x = 1$; the tip of the new parabola has y-coordinate $y = \frac{1}{4} \cdot 2 = \frac{1}{2}$. With this information and the knowledge that the parabola opens upward (because the coefficient of x^2 is positive), it is pretty easy to sketch the graph.



I cheated just a bit and let software do it for me ... :-) ■

Quiz #4. Tuesday, 24 May, 2016. *[10 minutes]*

Let $f(x) = x - 1$ and $g(x) = 10^x + 2$.

1. Find expressions in terms of x for $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$. *[2]*

2. Find expressions in terms of x for $f^{-1}(x)$, $g^{-1}(x)$, and $(g \circ f)^{-1}(x)$. *[3]*

SOLUTIONS. 1. To find the expression for $(f \circ g)(x) = f(g(x))$, we substitute the expression for $g(x)$ in for x in the expression for $f(x)$:

$$(f \circ g)(x) = f(g(x)) = (f \circ g)(x) = f(10^x + 2) = [10^x + 2] - 1 = 10^x + 1$$

Similarly, to find the expression for $(g \circ f)(x) = g(f(x))$, we substitute the expression for $f(x)$ in for x in the expression for $g(x)$:

$$(g \circ f)(x) = g(f(x)) = (g \circ f)(x) = g(x - 1) = 10^{x-1} + 2 \qquad \square$$

2. To find the expression for $f^{-1}(x)$, we try to solve the equation $y = f(x)$ for x:

$$y = f(x) = x - 1 \Rightarrow x = y + 1$$

Interchanging the roles of x and y, it follows that $f^{-1}(x) = x + 1$. Similarly, to find the expression for $g^{-1}(x)$, we try to solve the equation $y = g(x)$ for x:

$$y = g(x) = 10^x + 2 \Rightarrow 10^x = y - 2 \Rightarrow x = \log(10^x) = \log(y - 2)$$

[Recall that log without a specified base usually means $\log_{10}$.] Interchanging the roles of x and y, it follows that $g^{-1}(x) = \log(x - 2)$.

To find the expression for $(g \circ f)^{-1}(x)$, we present a slightly different way to do the book-keeping: let $x = (g \circ f)(y)$ and solve for $y = (g \circ f)^{-1}(x)$. That is, we interchange the roles of x and y first, rather than last. Using part of the answer to question 1, we then have:

$$\begin{aligned} x = (g \circ f)^{-1}(y) = 10^{y-1} + 2 &\Rightarrow 10^{y-1} = x - 2 \\ &\Rightarrow y - 1 = \log\left(10^{y-1}\right) = \log(x - 2) \\ &\Rightarrow y = \log(x - 2) + 1 \end{aligned}$$

Thus $(g \circ f)^{-1}(x) = \log(x - 2) + 1$. ■

NOTE. We can use the fact that $h^{-1}(h(x))$ is supposed to equal x to check our answers for question 2 if we have the time and energy. For example,

$$\begin{aligned} (g \circ f)^{-1}((g \circ f)(x)) &= (g \circ f)^{-1}\left(10^{x-1} + 2\right) = \log\left(\left[10^{x-1} + 2\right] - 2\right) + 1 \\ &= \log\left(10^{x-1}\right) + 1 = (x - 1) + 1 = x\,, \end{aligned}$$

so the answer we obtained above is correct. (Unless we've made another mistake ... :-)

Quiz #5. Thursday, 26 May, 2016. *[10 minutes]*

Recall from class that $\sin(30°) = \cos(60°) = \frac{1}{2}$, $\sin(45°) = \cos(45°) = \frac{1}{\sqrt{2}}$, and $\sin(60°) = \cos(30°) = \frac{\sqrt{3}}{2}$. Use these values of $\sin(x)$ and $\cos(x)$ and appropriate facts about trigonometric functions to help compute the following:

1. $\sin(120°)$ and $\cos(120°)$. *[2]*
2. $\tan(225°)$. *[1.5]*
3. $\tan^2(-135°) - \sec^2(-135°)$. *[1.5]*

NOTE: Work it out in each case – do not just plug angles into a calculator, except to check your answers.

SOLUTIONS. 1. Since $120° = 2 \cdot 60°$, we can use the double-angle formulas for $\sin(x)$ and $\cos(x)$. First:

$$\sin(120°) = \sin(2 \cdot 60°) = 2\sin(60°)\cos(60°) = 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}$$

Second:

$$\cos(120°) = \cos(2 \cdot 60°) = \cos^2(60°) - \sin^2(60°) = \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$$

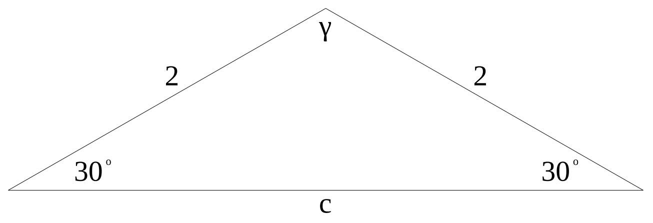
Note that one could do this question in several other ways. For example, one could use the fact that $(\cos(120°), \sin(120°))$ is the reflection of $(\cos(60°), \sin(60°))$ in the y-axis. [Why is it?] □

2. By definition, $\tan(225°) = \dfrac{\sin(225°)}{\cos(225°)}$, so we need to determine $\sin(210°)$ and $\cos(210°)$. Since $225° = 180° + 45°$, the point $(\cos(225°), \sin(225°))$ is the opposite point on the unit circle from the point $(\cos(45°), \sin(45°)) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. It follows that $\cos(225°) = \sin(225°) = -\frac{1}{\sqrt{2}}$, so $\tan(225°) = \frac{-1/\sqrt{2}}{-1/\sqrt{2}} = 1$. Of course, this question could also be done in several other ways ... □

3. The hard ways to do this involve working it out using the definitions of $\tan(x)$ and $\sec(x)$ in terms of $\sin(x)$ and $\cos(x)$. A better approach is to notice that the trigonometric identity $1 + \tan^2(x) = \sec^2(x)$ can be rearranged to give $\tan^2(x) - \sec^2(x) = -1$, which works for all x for which $\tan(x)$ and $\sec(x)$ are defined, *i.e.* for all x for which $\cos(x) \neq 0$. Since $\cos(x) = 0$ only when $x = 90° \pm n \cdot 180°$ for some integer n, that isn't a problem here, and so $\tan^2(-135°) - \sec^2(-135°) = -1$. ■

Quiz #6. Thursday, 2 June, 2016. *[10 minutes]*

Consider the following triangle:



1. Find the angle γ of the triangle. *[1]*
2. Find the length of the side c of the triangle. *[4]*

SOLUTIONS. 1. Since the sum of the interior angles of any (Euclidean) triangle is $180°$, $\gamma + 30° + 30° = 180°$, so $\gamma = 180° - 30° - 30° = 120°$. □

2. Since we now have $\gamma = 120°$ from question 1, we can apply the Law of Cosines:

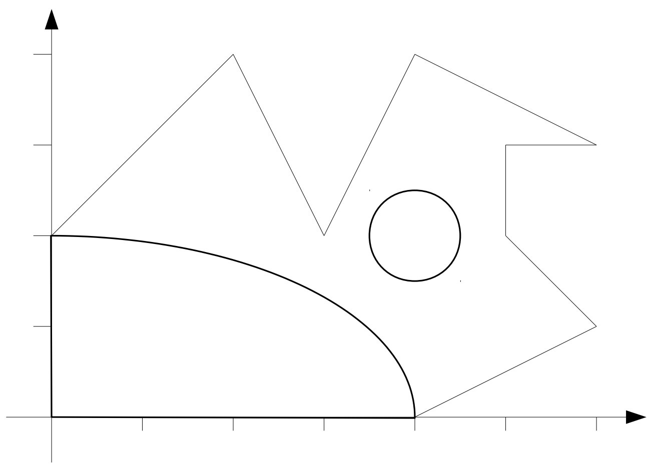
$$c^2 = 2^2 + 2^2 - 2 \cdot 2 \cdot 2 \cdot \cos(120°) = 4 + 4 - 8\left(-\frac{1}{2}\right) = 8 - (-4) = 8 + 4 = 12$$

Note that $\cos(120°) = -\cos(60°) = -\frac{1}{2}$. (Because?) It follows that $c = \sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$. (We need to take the positive root here because a side of a triangle cannot have negative length.) ■

Quiz #7. Tuesday, 2 June, 2016. *[10 minutes]*

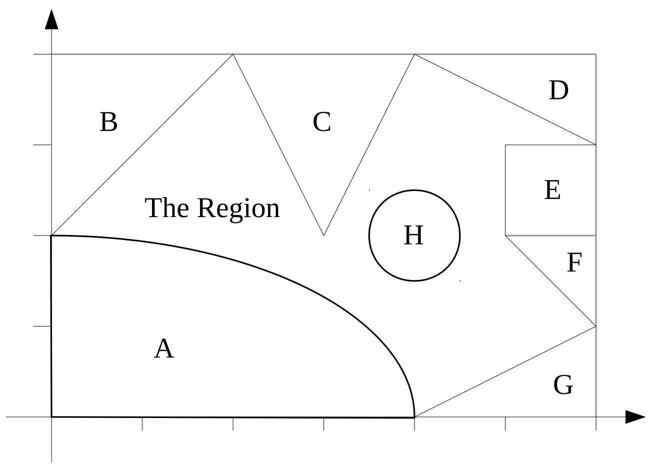
Consider the region whose border consists of [going counterclockwise from $(0,2)$]:

- the part of the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$ for which $x \geq 0$ and $y \geq 0$,
- the part of the line $y = \frac{1}{2}x - 2$ where $4 \leq x \leq 6$,
- the part of the line $y = -x + 7$ where $5 \leq x \leq 6$,
- the part of the line $x = 5$ for which $2 \leq y \leq 3$,
- the part of the line $y = 3$ for which $5 \leq x \leq 6$,
- the part of the line $y = -\frac{1}{2}x + 6$ where $4 \leq x \leq 6$,
- the part of the line $y = 2x - 4$ where $3 \leq x \leq 4$,
- the part of the line $y = -2x + 8$ where $2 \leq x \leq 3$, and
- the part of the line $y = x + 2$ where $0 \leq x \leq 2$, and also
- the circle $(x-4)^2 + (y-2)^2 = \frac{1}{4}$. [The circle makes a hole in the region.]



1. Compute the area of the given region. *[5]*

Solution. Consider the diagram below:



The smallest rectangle containing the given region has vertices at $(0,0)$, $(0,4)$, $(6,4)$, and $(6,0)$, as in the diagram; this rectangle has area $6 \cdot 4 = 24$.

The given region is what is left after the shapes labelled A through H are removed from the rectangle, so the area of the region is the area of the rectangle minus the areas of these shapes, which we compute below:

A. The ellipse $\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$ has an area of $4 \cdot 2 \cdot \pi = 8\pi$. Since A is one quarter of this ellipse, it has area $\frac{1}{4} \cdot 8\pi = 2\pi$.

B. B is a triangle with base and height 2, so it has area $\frac{1}{2} \cdot 2 \cdot 2 = 2$.

C. C is another triangle with base and height 2, so it has area $\frac{1}{2} \cdot 2 \cdot 2 = 2$.

D. D is a triangle with base 2 and height 1, so it has area $\frac{1}{2} \cdot 2 \cdot 1 = 1$.

E. E is a square with sides of length 1, so it has area $1^2 = 1$.

F. F is a triangle with base and height 1, so it has area $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$.

G. G is another triangle with base 2 and height 1, so it has area $\frac{1}{2} \cdot 2 \cdot 1 = 1$.

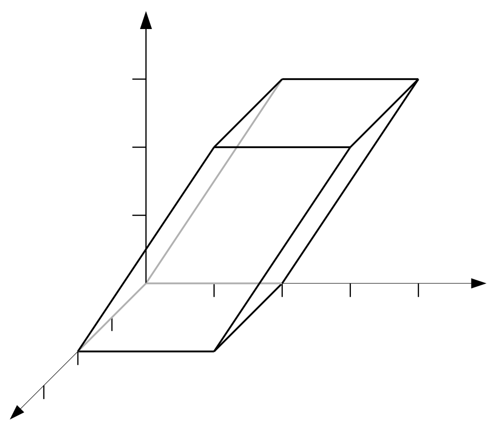
H. H is a circle with radius $\sqrt{\frac{1}{4}} = \frac{1}{2}$, so it has area $\pi \cdot \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$.

Thus the area of the given region is:

$$24 - 2\pi - 2 - 2 - 1 - 1 - \frac{1}{2} - 1 - \frac{\pi}{4} = \frac{33}{5} - \frac{9}{4}\pi \approx 9.43142 \qquad \blacksquare$$

Quiz #8. Thursday, 9 June, 2016. *[10 minutes]*

A rectangular box of length 2, width 2, and height 3 is distorted by sliding the top face 2 units to the right (so its left edge ends up exactly where its right edge used to be).

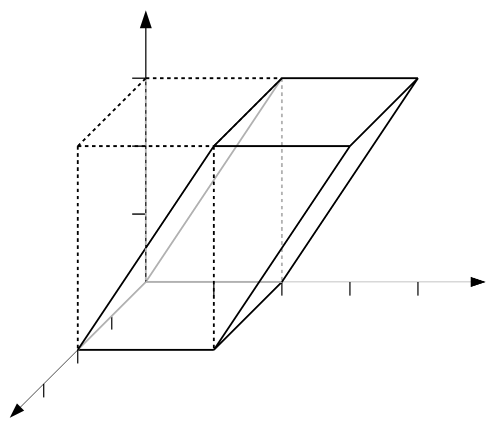


1. Find the volume of the distorted box. *[2]*
2. Find the surface area of the distorted box. *[3]*

Solutions. 1. *(One way.)* The distorted box has exactly the same horizontal cross-sections as the original box, just arranged a little differently. It follows by Cavalieri's Principle that it has the same volume as the original rectangular box: $2 \cdot 2 \cdot 3 = 12$. $\square$

1. *(Another way.)* Notice that the distorted box has the left edge of the top face directly over the right edge of the bottom face. (The distortion was done by sliding the the top

face to the right until the left edge of the top face was exactly where the right edge of the top face used to be, and that was right over the right edge of the bottom face.) If you cut the distorted box vertically from the top left edge to the bottom right edge, you get two triangular prisms; sliding the right one left two units gives us back the original box.



It follows that the volume of the distorted box is equal to the volume of the original box: $2 \cdot 2 \cdot 3 = 12$. □

2. The top and bottom faces of the distorted box are squares with sides of length 2, and so each has area $2 \cdot 2 = 4$.

The front and back faces of the box can each be cut into two right triangles with base 2 and height 3 (see the reasoning and diagram in the *Another way* solution to question 1), so each face has area $2 \cdot \frac{1}{2} \cdot 2 \cdot 3 = 6$.

The left and right faces are rectangles of width 2 and length equal to the hypotenuse of a right triangle with short sides of 2 and 3, respectively (again, look at that diagram), which length is $\sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$. Each of these faces therefore has area $2 \cdot \sqrt{13}$.

Adding up the areas of the faces gives us the surface area of the distorted box:

$$[4+4] + [6+6] + \left[2\sqrt{13} + 2\sqrt{13}\right] = 20 + 4\sqrt{13} \approx 34.42$$

Note that this is not equal to the surface area of the original rectangular box, which is $2 \cdot 4 + 4 \cdot 6 = 8 + 24 = 32$. (The difference is because the left and right faces of the original box are 2×3 rectangles.) ■

Quiz #9. Tuesday, 14 June, 2016. *[10 minutes]*

Suppose we are given a sequence a_k such that $a_1 = 8$ and $a_4 = 64$.

1. Assuming that the sequence is arithmetic, find the initial term $a_0 = a$ and common difference d. *[1.5]*
2. Assuming that the sequence is geometric, find the initial term $a_0 = a$ and the common ratio r. *[1.5]*
3. Compute $4 + 8 + 16 + 32 + \cdots + 2048 + 4096$. *[2]*

SOLUTIONS. 1. In general, we have $a_k = a + kd$ for an arithmetic sequence, which means that $8 = a_1 = a+d$ and $64 = a_4 = a+4d$ in this case. It follows that $3d = (a+4d)-(a+d) = 64 - 8 = 56$, so $d = \frac{56}{3}$. Plugging this back in to the equation for a_1 gives us $8 = a + \frac{56}{3}$, so $8 - \frac{56}{3} = \frac{24}{3} - \frac{56}{3} = -\frac{32}{3} \approx -10.67$. □

2. In general, we have $a_k = ar^k$ for a geometric sequence, which means that $8 = a_1 = ar$ and $64 = a_4 = ar^4$. It follows that $r^3 = \frac{ar^4}{ar} = \frac{64}{8} = 8$, so $r = 8^{1/3} = 2$. Plugging this back into the equation for a_1 gives us $8 = a \cdot 2$, so $a = \frac{8}{2} = 4$. □

3. 4, 8, 16, 32, ... , 2048, 4096 is a geometric sequence with initial term $a = 4$ and common ratio $r = 2$. Since $4096 = 4 \cdot 1024 = 4 \cdot 2^{10}$, we have all we need to apply the formula for the sum of a finite geometric series, *i.e.* $a + ar + ar^2 + \cdots + ar^k = a \cdot \frac{1-r^{k+1}}{1-r}$:

$$\begin{aligned} 4 + 8 + 16 + 32 + \cdots + 2048 + 4096 &= 4 \cdot \frac{1 - 2^{10+1}}{1-2} = 4 \cdot \frac{1-2048}{-1} \\ &= 4(2048 - 1) = 4 \cdot 2047 = 8188 \qquad \blacksquare \end{aligned}$$

Quiz #10. Thursday, 16 June, 2016. *[10 minutes]*

Do *one* of the following two questions.

1. How many ways are there to select a chairperson and two other members of a committee from a pool of ten candidates? *[5]*
2. What does $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n}$ equal? *[5]*

SOLUTIONS. 1. There are $\binom{10}{1} = 10$ ways to choose a chairperson from the pool and, once that is done, $\binom{9}{2} = 36$ ways to choose the other two committee members from the nine remaining candidates. This means that there are $\binom{10}{1}\binom{9}{2} = 10 \cdot 36 = 360$ ways to pick a chairperson and two other committee members from a pool of ten candidates. □

2. Plugging $x = -1$ into $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$ gives:

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} = (1 + (-1))^n = 0^n = 0 \ \blacksquare$$