Complex Analysis. Lecture notes

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These lecture notes cover undergraduate course in Complex Analysis that was taught at Trent Univesity at 2006-2007.

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Chapter 1

Complex numbers

1.1 Definitions

By convention, $\mathbf{R}^n$ is the vector space of real vector columns $(x_1, ..., x_n)^\top$, where $x_i \in \mathbf{R}$.

We know few classes of numbers: the set of integers $\mathbf{Z} = \{0, \pm 1, \pm 2, ...\}$, the set of rational numbers, the set of real numbers $\mathbf{R} = (-\infty, \infty)$.

Now we have a new class: complex numbers. Let i be some fixed symbol (we shall call it "imaginary unit"). Assume that any vector $(x, y) \in \mathbf{R}^2$ is represented in the form $x + iy$. We shall call this form a complex number. We assume that any real number is also a complex number: $x = x + 0 \cdot i$.

Let $z = x + iy$ be a complex number, $x, y \in \mathbf{R}$. x is said to be the real part $\operatorname{Re} z$ of z, and y is said to be the imaginary part $\operatorname{Im} z$ of z. Real numbers are placed on the so-called real axes, and complex numbers are being placed on the so-cable imaginary axes.

1.2 Module and argument

Let $z = x + iy$ be a complex number, $x, y \in \mathbf{R}$.

Definition 1.1 *The module $|z|$ of z is*

$$|z| = \sqrt{x^2 + y^2} = \sqrt{\operatorname{Re} z^2 + \operatorname{Im} z^2}.$$

$|z|$ is the distance from z to the zero.

Definition 1.2 *The argument $\arg z$ of $z \in \mathbf{C}$, $z \neq 0$, is the angle (in radians) between the arrow directed to z and the real axis.*

For instance, if $z = z + iy$, $x > 0$, then

$$\arg z = \arctan \frac{y}{x} + 2\pi k, \quad k = 0, \pm 1, \pm 2....$$

Note that the angle is not unique, since

$$\cos\alpha = \cos(\alpha + 2\pi k), \quad \sin\alpha = \sin(\alpha + 2\pi k), \quad \tan\alpha = \tan(\alpha + 2\pi k).$$

The version of the argument in $(-\pi, \pi]$ is said to be the main (or principal) value of $\arg z$, and it is denoted as $\operatorname{Arg} z$.[1]

C is the standard notation for the set of complex numbers.

1.3 Addition and multiplication

Definition 1.3 *We define addition and multiplication as the following: for* $z_k = x_k + iy_k$*, where* $x_k, y_k \in \mathbf{R}$*,*

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2),$$
$$z_1 \cdot z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1).$$

In particular, $i^2 = i \cdot i = -1 + i \cdot 0 = -1$. Therefore, we have that the equation $z^2 = -1$ is solvable!

In fact, this means that the set $\mathbf{R}^2$ is provided with the standard addition (as in the vector space $\mathbf{R}^2$) and with the special multiplication.

For $z = x + iy$, we denote $-z = (-1)z = (-1 - i \cdot 0)z = -x - iy$. We denote $0 = 0 + i \cdot 0$. We have $z \cdot 0 = 0 \cdot z = 0$ for all $z \in \mathbf{C}$.

In addition, we assume that $z_1 - z_2 = z_1 + (-z_2)$.

Inversion

Let $z \in \mathbf{C}$, then z^{-1} is a number such that $z \cdot z^{-1} = 1$. In fact, it exists and it is uniquely defined for all $z \neq 0$. We assume also that $z_1/z_2 = z_1 z_2^{-1}$.

Triangle inequality

Note that $|z_1 - z_2|$ is the distance between z_1 and z_2 in $\mathbf{R}^2$. Therefore, it is easy to see that the following triangle inequality holds:

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Proof. Let $\omega \triangleq -z_2$, then $|z_1 - \omega| \leq |z_1| + |-\omega|$, by the property of the distance. □

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[1] In literature, the main or principal value of $\arg z$ is sometimes defined differently as a version of the argument from $[0, 2\pi)$.

1.3.1 Conjugate numbers

Let $z \in \mathbf{C}$, $z = x + iy$, where $x, y \in \mathbf{R}$. The number $\overline{z} \triangleq x - iy$ is said to be conjugate (with respect to z). Note that $z \cdot \overline{z} = x^2 + y^2 = |z|^2$. It is a real nonnegative number.

1.3.2 How to calculate $1/z$

We have

$$\frac{1}{z} = \frac{1}{z}\frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}.$$

For instance,

$$\frac{1}{3+2i} = \frac{3-2i}{9+4} = \frac{3}{13} - \frac{2}{13}i.$$

1.4 Polar form form of a complex number

Let $x, y, r, \varphi \in \mathbf{R}$, $z = x + iy$, $r = |z|$, $\varphi = \arg z$, then

$$x = r\cos\varphi, \quad y = r\sin\varphi,$$

i.e., $z = r(\cos\varphi + i\sin\varphi)$.

1.4.1 Multiplication in the polar form

Let $z_k = x_k + iy_k$, $k = 1, 2$, $r_k = |z_k|$, $\varphi_k = \arg z_k$. Let $z = z_1 z_2$. We have

$$\begin{aligned} z &= x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2) \\ &= r_1r_2(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 + i[\cos\varphi_1\sin\varphi_2 + \sin\varphi_1\cos\varphi_2]) \\ &= r_1r_2(\cos(\varphi_1+\varphi_2) - i\sin(\varphi_1+\varphi_2)). \end{aligned}$$

It follows that

$$|z_1z_2| = |z_1||z_2|, \qquad \arg(z_1z_2) = \arg z_1 + \arg z_2.$$

Corollary 1.4 *If $r, \varphi \in \mathbf{R}$, $r > 0$, $z = r(\cos\varphi + i\sin\varphi)$, then*

$$z^m = r^m(\cos(m\varphi) + i\sin(m\varphi)), \quad m = 1, 2, 3, ...$$

Corollary 1.5 *If $r, \varphi \in \mathbf{R}$, $r > 0$, $z = r(\cos\varphi + i\sin\varphi)$, then*

$$z^{-1} = r^{-1}(\cos(-\varphi) + i\sin(-\varphi)).$$

1.5 Roots from a complex number

Let $\omega, z \in \mathbf{C}$ be such that $\omega^m = z$, $m \in \{2,3,4,...\}$. We say ω is a root of order m from z.

Let $r, \varphi \in \mathbf{R}$, $r > 0$, $z = r(\cos\varphi + i\sin\varphi)$, $m \in \{2,3,4,...\}$. Let

$$\omega_k \triangleq r^{1/m}\left(\cos\theta_k + i\sin\theta_k\right), \quad k = 0,1,2,3,...,m-1,$$

where $\theta_k = \frac{\varphi+2\pi k}{m}$. We have that

$$\omega_k^m = r(cos\varphi + i\sin\varphi) = z.$$

Therefore, z has at least m different complex roots of order m (it will be seen later that there are exactly m roots).

For example, this works for $z = 1$: in our notations, $\omega_0 = 1$, $\omega_1 = -1$. Similarly, for any $z \in \mathbf{C}$, we have that if $\omega^2 = z$, then $(-\omega)^2 = z$.

1.5.1 Quadratic equation

Consider equation $z^2 + pz + q = 0$, where $p, q \in \mathbf{C}$. Let ω be any square root from $D \triangleq p^2/4 - q$. Let $z_1 = -p/2 - \omega$, $z_2 = -p/2 + \omega$. It can be verified immediately that

$$(z - z_1)(z - z_2) = z^2 + pz + q.$$

Hence z_k are (the only) roots of this equation.

1.5.2 The Fundamental Theorem of Algebra

Theorem 1.6 *Any polynomial of order $n \in \{1,2,3...\}$*

$$P(z) = z^n + c_{n-1}z^{n-1} + ... + c_1 z + c_0,$$

where $c_k \in \mathbf{C}$, has n roots in $\mathbf{C}$, i.e. it can be presented as

$$P(z) = (z - z_1)\cdot(z - z_2)\cdots(z - z_n)$$

for some $z_k \in \mathbf{C}$, $k = 1,2,...,n$.

Proof will be given later.

Note that it is a difficult problem to find the roots of a polynomial explicitly if $n > 3$.

Chapter 2

Elements of analysis

2.1 Limits and convergence

Let $\{z_k\} \subset \mathbf{C}$ be a sequence, and let $z \in \mathbf{C}$.

Definition 2.1 *We say that $z_i \to z$ (in $\mathbf{C}$) as $k \to +\infty$ (i.e., $z = \lim_k z_k$) iff $|z_k - z| \to 0$.*

Lemma 2.2 *$z_i \to z$ (in $\mathbf{C}$) as $k \to +\infty$ iff $\operatorname{Re} z_i \to \operatorname{Re} z$ and $\operatorname{Im} z_i \to \operatorname{Im} z$.*

In other words, this convergence is the same as the convergence in $\mathbf{R}^2$ (with Euclidean norm) for the vector consisting of the real and imaginary parts.

Definition 2.3 *We say that $z_k \to \infty$ as $k \to +\infty$, if $|z_k| \to +\infty$.*

Note that ∞ and $+\infty$ have different meaning in the definition above.

2.2 Series

Let $\{z_k\} \subset \mathbf{C}$ be a sequence. Let $\{c_k\}$ be the sequence of the partial sums:

$$\begin{aligned} c_1 &= z_1, \\ c_2 &= z_1 + z_2, \\ &\dots \\ c_n &= z_1 + ... + z_n, \\ &\dots \end{aligned}$$

Definition 2.4 *We say that series $z_1 + z_2 + z_3 + \dots$ converges if the sequence $\{c_k\}$ of the partial sums has a limit in $\mathbf{C}$. This limit is said to be the summa of the series. In other words,*

$$\sum_{k=1}^{+\infty} z_k = \lim_k c_k.$$

Definition 2.5 *We say that series* $z_1 + z_2 + z_3 + ...$ *absolutely converges if the series* $|z_1| + |z_2| + |z_3| + ...$ *converges.*

Theorem 2.6 *If a series* $z_1 + z_2 + z_3 + ...$ *absolutely converges then this series converges.*

Proof. It follows from the properties of convergence of the series in $\mathbf{R}$ (or even in $\mathbf{R}^2$). For instance, it can be seen that the sequences $\{\operatorname{Re} z_k\}$ and $\{\operatorname{Im} z_k\}$ absolutely converge, therefore they converge and have limits. □

2.2.1 Power series

Let $\{c_k\} \subset \mathbf{C}$ be a sequence, $a \in \mathbf{C}$. A series in the form

$$c_0 + c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + ...$$

is said to be power series.

Example: we have

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + ...$$

This series converge for any z such that $|z| < 1$ (it follows from the fact that the series absolutely converges).

Definition 2.7 *Given a power series* $\sum_k c_k(z-a)^k$, *the radius of convergence is defined as*

$$\sup\{|z-a| : \sum_k |c_k(z-a)^k| \quad \textit{converges}\}.$$

2.3 Exponent

Remind that

$$\begin{aligned} &e^t = 1 + t + \tfrac{t^2}{2!} + \tfrac{t^3}{3!} + ..., \\ &e^{s+t} = e^s e^t, \\ &(e^t)' = e^t, \quad (e^{at})' = ae^{at}, \quad t \in \mathbf{R} \end{aligned}$$

Definition 2.8 *Let* $z \in \mathbf{C}$. *We define* e^z *as*

$$e^z \triangleq 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$$

Note that the series in the definition above converges for all z since this series is absolutely converges:

$$1 + |z| + \frac{|z|^2}{2!} + \frac{|z|^3}{3!} + ... = e^{|z|}$$

Lemma 2.9 *For all $a, b \in \mathbf{C}$,*

$$e^{a+b} = e^a e^b.$$

Proof.

$$\begin{aligned}
e^a e^b &= \left(\sum_{k=0}^{+\infty} \frac{a^k}{k!}\right)\left(\sum_{h=0}^{+\infty} \frac{b^h}{h!}\right) = \sum_{k=0}^{+\infty}\sum_{h=0}^{+\infty} \frac{a^k b^h}{k!h!} = \sum_{k=0}^{+\infty}\sum_{h=0}^{k} \frac{a^{k-h}b^h}{(k-h)!h!} \\
&= \sum_{k=0}^{+\infty} \frac{1}{k!} \sum_{h=0}^{k} \frac{k! a^{k-h} b^h}{(k-h)!h!} = \sum_{k=0}^{+\infty} \frac{1}{k!}(a+b)^k = e^{a+b}.
\end{aligned}$$

2.3.1 Euler's formula

Theorem 2.10 *(Euler's formula): If $z = x + iy$, $x, y \in \mathbf{R}$, then*

$$e^z = e^x(\cos y + i \sin y).$$

We have $e^z = e^x e^{iy}$. To explore the form of e^z, it suffices to study e^{iy} for real y. We have

$$\begin{aligned}
e^{iy} &= \sum_{k=0}^{+\infty} \frac{(iy)^k}{k!} = \sum_{k=2m,\ m=0,1,2,..} \frac{(iy)^k}{k!} + \sum_{k=2m+1,\ m=0,1,2,..} \frac{(iy)^k}{k!} \\
&= \sum_{m=0,1,2,..} (-1)^m \frac{y^{2m}}{(2m)!} + i \sum_{m=0,1,2,..} (-1)^m \frac{y^{2m+1}}{(2m+1)!} \\
&= \cos y + i \sin y.
\end{aligned}$$

This completes the proof.

2.3.2 Parametrization of a circle

Let $\omega \in [0, 2\pi)$, then the values of e^{iw} form the unit circle.

2.3.3 Differentiation with respect to a real variable

Let $f : \mathbf{R} \to \mathbf{C}$, i.e., $f(t) = a(t) + ib(t)$, where $a(\cdot), b(\cdot)$ are real functions. Similarly to differentiation of a vector function, we assume that $f'(t) = (a(t))' + i(b(t))' = (\operatorname{Re} f(t))' + i(\operatorname{Im} f(t))'$.

Let $y(t) = e^{at}$, $a = x + iy \in \mathbf{C}$, $x, y, t \in \mathbf{R}$. We have that

$$\begin{aligned}
\frac{dy}{dt}(t) &= (e^{xt}\cos(yt))' + i(e^{xt}\sin(yt) \\
&= xe^{xt}\cos(yt) - ye^{xt}\sin(yt) + i[xe^{xt}\sin(yt) + ye^{xt}\cos(yt)] \\
&= (x + iy)e^{xt}[\cos(yt) + i\sin(yt)] = ae^{at} = ay(t).
\end{aligned}$$

2.3.4 An application: solution of an ordinary differential equation

Let us consider a second order ODE

$$y''(t) + py'(t) + qy = 0. \tag{2.1}$$

Let $\lambda_{1,2} = -p/2 \pm W$, where W is a square root from $p^2/4 - q$. We saw that λ_k are roots of the equation

$$\lambda^2 + p\lambda + q = 0.$$

Let

$$y_k(t) \triangleq e^{\lambda_k t}, \quad k = 1, 2.$$

Then

$$y_k''(t) + py_k'(t) + qy = e^{\lambda_k t}(\lambda_k^2 + p\lambda_k + q) = 0.$$

Therefore, $y_k(t)$ are solutions of the ODE. By the linearity, it follows that any process

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{2.2}$$

is also a solution, for any $c_1, c_2 \in \mathbf{C}$. (In ODE courses, it is being proved that any solution of (2.1) can be represented in this form if $\lambda_1 \neq \lambda_2$; we omit this part).

For the most interesting cases, p, q are real, and one is interested in real solutions.

Problem 2.11 *Prove that if p, q are real, and* $\mathrm{Im}\, c_1 = -\mathrm{Im}\, c_2$, $\mathrm{Re}\, c_1 = \mathrm{Re}\, c_2$, *then the process (2.2) is real.*

If $\lambda_k = r \pm i\omega$, $r, \omega \in \mathbf{R}$, then the real solutions can be represented in the form

$$y(t) = e^{rt}(C_1 \cos(\omega t) + C_2 \sin(\omega t)), \quad C_1, C_2 \in \mathbf{R}.$$

In that case, $y(t)$ is an oscillating process with decay/growth e^{rt}; ω is referred as the frequency.

2.4 Other elementary functions

We have defined already functions z^m, for $m = \pm 1, \pm 2, \pm 3, \ldots$. We introduce below few more elementary functions.

2.4.1 cos and sin

We define

$$\cos z \triangleq \sum_{m=0,1,2,..} (-1)^m \frac{z^{2m}}{(2m)!}, \quad \sin z \triangleq \sum_{m=0,1,2,..} (-1)^m \frac{z^{2m+1}}{(2m+1)!}.$$

It can be seen that $\cos z = (e^{iz} + e^{-iz})/2$, $\sin z = (e^{iz} - e^{-iz})/(2i)$.

2.4.2 Logarithm

Let $x > 0$, then $y = \ln x$ is such that $e^y = x$.We require that $x > 0$ because $e^y > 0$ for all $y \in \mathbf{R}$.

If $z \in \mathbf{C}$, then e^z is not "positive", it is a complex number (for the general case).

However, we are going to define $\log z$ for all $z \neq 0$ as the inverse of the exponent. Set

$$\log z \triangleq \log |z| + i\mathrm{arg}\, z.$$

Note that this value is not unique, since $\arg z$ is not unique.

It is easy to see that

$$e^{\log z} = e^{\log |z|}(\cos \arg z + i \sin(\arg z)) = z.$$

Convention

Recall that we assume that $\mathrm{Arg}\, z \in (-\pi, \pi]$. We denote as $\mathrm{Log}\, z$ the corresponding value of $\log z$, i.e., $\mathrm{Log}\, z = \ln |z| + i\mathrm{Arg}\, z$.

2.5 Continuity and differentiability. Holomorphic functions

Definition 2.12 *We say that $D \subset \mathbf{C}$ is an open set iff for any point $x \in D$ there exists $\varepsilon > 0$ such that $\{y \in \mathbf{C} : |x - y| \leq \varepsilon\} \subset D\}$.*

Let $D \subset \mathbf{C}$ be an open set, $f : D \to \mathbf{C}$ be a function.

Definition 2.13 *We say that f is continuous at $z \in D$ if, for all $z \in \mathbf{C}$, $\{z_k\} \subset D$,*

$$z_k \to z \quad \text{as } k \to +\infty \quad \Rightarrow \quad f(z_k) \to f(z) \quad \text{as } k \to +\infty.$$

We say that f is continuous on D if f is continuous at all $z \in D$.

Definition 2.14 *We say that f is differentiable at $z \in D$ iff there exists a number $f'(z) \in \mathbf{C}$ such that, for all $\{\Delta_k\} \subset \mathbf{C}$, such that $z + \Delta_k \in D$, $\Delta_k \neq 0$, we have that*

$$\Delta_k \to 0 \quad \text{as } k \to +\infty \quad \Rightarrow \quad \left| \frac{f(z + \Delta_k) - f(z)}{\Delta_k} - f'(z) \right| \to 0 \quad \text{as } k \to +\infty.$$

The value $f'(z)$ is said to be the (first) derivative of f at z (it is denoted also as $df(z)/dz$).

It can be written as

$$f(z + \Delta_k) - f(z) = f'(z)\Delta_k + o(\Delta_k),$$

or

$$\frac{f(z+\Delta_k)-f(z)}{\Delta_k}=f'(z)+O(\Delta_k).$$

Here we use the popular and commonly used notations $o(\cdot)$ and $O(\cdot)$ for the remainders: $o(z)$ and $O(z)$ are some functions such that $o(z)/z \to 0$ and $O(z) \to 0$ as $z \to 0$; these terms are used for convenience.

Definition 2.15 *We say that f is holomorphic* [1] *in D if f is differentiable at every point of D.*

Lemma 2.16 *If f is differentiabe at z, then f is continuous at z.*

Corollary 2.17 *If f is holomorphic in D, then f is continuous in D.*

2.5.1 Example of non-differentiability

In fact, the definition of differentiability is more restrictive than it looks, since Δ_k in this definition is allowed to converge to zero via any path. For instance, this definition ensures that the function $f(z) = \mathrm{Re}\, z$ is non-differentiable. Let us show this.

Let $\Delta_k = x_k + iy_k$, where $x_k, y_k \in \mathbf{R}$.

Let $z = 0$, $y_k \equiv 0$, then

$$\frac{f(z+\Delta_k)-f(z)}{\Delta_k}=\frac{\mathrm{Re}\,(0+x_k)-\mathrm{Re}\,(0)}{x_k}\equiv 1.$$

On the other hand, if $x_k \equiv 0$, then

$$\frac{f(z+\Delta_k)-f(z)}{\Delta_k}=\frac{\mathrm{Re}\,(0+iy_k)-\mathrm{Re}\,(0)}{iy_k}\equiv 0.$$

$\square$

Problem 2.18 *Show that the functions $f(z) = \mathrm{Im}\, z$, $f(z) = \overline{z}$ are non-differentiable.*

2.6 Basic derivatives

2.6.1 Power functions

Let $m \in \{1, 2, 3, ...\}$.

Lemma 2.19 $(z^m)' = mz^{m-1}$.

[1] In the literature, the functions from Definition 2.15 are often referred as the analytic functions. We will define analytic functions differently, and we will show that these definitions are equivalent.

2.6.2 Exponent

Lemma 2.20 $(e^z)' = e^z$.

Lemma 2.21 *Let $a \in \mathbf{C}$ be given, then $(e^{az})'_z = ae^{az}$.*

2.6.3 Inversion

Let $z \neq 0$, then

$$\left(\frac{1}{z}\right)' = -\frac{1}{z^2}$$

2.6.4 Derivative of a product

Lemma 2.22 *If $f(\cdot)$ is differentiable at z, and $g(z)$ is differentiable at z, then $F(z) \triangleq f(z)g(z)$ is differentiable at z, and $F'(z) = f'(z)g(z) + f(z))g'(z)$.*

Clearly, $(\alpha)' = 0$ for any constant $\alpha \in \mathbf{C}$. It follows that $(\alpha f(z))' = \alpha f'(z)$, for any $\alpha \in \mathbf{C}$. In addition, it can be proved easily that $(f(z) + g(z))' = f'(z) + g'(z)$.

2.6.5 The chain rule

Lemma 2.23 *If $f(\cdot)$ is differentiable at z, and $g(\zeta)$ is differentiable at $\zeta = f(z)$, then $G(z) \triangleq g(f(z))$ is differentiable at z, and $G'(z) = g'(f(z))f'(z)$.*

Proof.

$$\begin{aligned} G(z+\Delta) - G(z) &= g(f(z+\Delta)) - g(f(z)) \\ &= g(f(z) + f'(z)\Delta + o(\Delta)) - g(f(z)) \\ &= g'(f(z))f'(z)\Delta + o(\Delta), \end{aligned}$$

since

$$f'(z)\Delta + o(\Delta) = 0(\Delta).$$

These rule help to find many other derivatives explicitly. For instance,

$$\left(\frac{1}{z-a}\right)' = -\frac{1}{(z-a)^2}.$$

2.7 The Cauchy-Riemann equations

Theorem 2.24 *(The Cauchy-Riemann equations). Let $f(z)$ be differentiable at $z = x+iy$, $x, y \in \mathbf{R}$. Let*

$$f(z) = f(x+iy) = u(x,y) + iv(x,y),$$

where $u : \mathbf{R}^2 \to \mathbf{R}$ and $v : \mathbf{R}^2 \to \mathbf{R}$ are real differentiable functions. Then

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y), \qquad \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y).$$

Proof. Let $f'(z) = A + iB$, where A and B are real.

Let $\Delta = \Delta x + i\Delta y$, where Δx and Δy are real. Then

$$f(z+\Delta) - f(z) = f'(z)\Delta + o(\Delta) = (A\Delta x - B\Delta y) + i(B\Delta x + A\Delta y) + o(\Delta).$$

Further,

$$\begin{aligned} u(x+\Delta x, y+\Delta y) - u(x,y) &= A\Delta x - B\Delta y + o(\Delta), \\ v(x+\Delta x, y+\Delta y) - v(x,y) = f'(z)\Delta + o(\Delta) &= B\Delta x + A\Delta y + o(\Delta). \end{aligned}$$

On the other hand,

$$\begin{aligned} u(x+\Delta x, y+\Delta y) - u(x,y) &= \frac{\partial u}{\partial x}(x,y)\Delta x + \frac{\partial u}{\partial y}(x,y)\Delta y + o(\Delta), \\ v(x+\Delta x, y+\Delta y) - v(x,y) &= \frac{\partial v}{\partial x}(x,y)\Delta x + \frac{\partial v}{\partial y}(x,y)\Delta y + o(\Delta). \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= A, \quad \frac{\partial u}{\partial y}(x,y) = -B, \\ \frac{\partial v}{\partial x}(x,y) &= B, \quad \frac{\partial v}{\partial y}(x,y) = A. \end{aligned}$$

Then the proof follows.

Corollary 2.25 *If u, v are twice differntiable, then*

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) &= 0, \\ \frac{\partial^2 v}{\partial x^2}(x,y) + \frac{\partial^2 v}{\partial y^2}(x,y) &= 0. \end{aligned}$$

It will be shown later that it follows from the differentiability that u, v are also twice differentiable. Therefore, by Corollary 2.25, both the imaginary and real parts must satisfy these partial differential equations. These particular equations are elliptic equations; they are called Laplace equations.

2.8 Antiderivative

Let $D \subset \mathbf{C}$ be an open domain, $f : D \to \mathbf{C}$ and $F : D \to \mathbf{C}$. We say that F is an antiderivative of f is $F'(z) = f(z)$ in D. Note that antiderivative is not unique ($F + \text{const}$ is also an antiderivative).

Chapter 3

Complex integration: path integrals

3.1 Curves

Definition 3.1 *Let $a, b \in \mathbf{R}$ be such that $a < b$. Let $\gamma : [a, b] \to \mathbf{C}$ be a continuous mapping, and let*

$$\Gamma \triangleq \{z \in \mathbf{C} : z = \gamma(t),\ t \in [a, b]\}.$$

We say that Γ is a curve in $\mathbf{C}$ (with the one-dimensional parametrization given by γ). If $\gamma(a) = \gamma(b)$, then we say that the curve is closed[1].

Note that Γ is a *connected set*. If $f : \mathbf{C} \to \mathbf{C}$ is a continuous function, then

$$f(\Gamma) \triangleq \{z \in \mathbf{C} : z = f(\gamma(t)),\ t \in [a, b]\}$$

is also a curve. If Γ is a closed curve, then the curve $f(\Gamma)$ is also closed.

Note that a set Γ may have many different one-dimensional parametrizations, and it is possible that $z \in \Gamma$ is such that $z = \gamma(t_1) = \gamma(t_2)$ for some $t_1 \neq t_2$.

Example 3.2 *Let $\gamma(t) = e^{it}$.*

(a) If $[a, b] = [0, 2\pi]$, then Γ is the circle, and it is a closed curve.

(b) If $[a, b] = [0, 4\pi]$, then Γ is the circle repeated twice; this curve is closed.

(b) If $[a, b] = [0, 3\pi]$, then Γ is the circle such that a half of it is repeated; this curve is not closed.

[1] In geometry and topology, there is a different term "closed set" based on the definition of the limit: a set A is said to be close iff it contains all its limit points.

3.2 Integral as the limit of Riemann sums

Let Γ be a curve (path) in $\mathbf{C}$ given parametrically via $\gamma : [a,b] \to \mathbf{C}$, i.e., $\Gamma = \{z = \gamma(t),\ t \in [a,b]\}$, where $\gamma : [a,b] \to \mathbf{C}$ is a mapping, $a, b \in \mathbf{R}$, $a < b$. Let $f : \mathbf{C} \to \mathbf{C}$ be a function. We say that the path integral of f along Γ is

$$\int_\Gamma f(z)dz = \lim_{N\to+\infty,\delta\to 0} \sum_{k=0}^{N-1} f(z_k)(z_{k+1} - z_k).$$

(This integral is also said to be a contour integral around the curve Γ, if Γ is closed.)

Here the limit is taken with respect to a choice of sets $\{z_k\}_{k=0}^N \subset \Gamma$ such that $N \to +\infty$, $\delta \to 0$, where $\delta \triangleq \max_k |z_{k+1} - z_k|$. We assume that the ponts z_k are placed consequently and $z_0 = \gamma(a)$, $z_N = \gamma(b)$. In other words, the set $\{z_k\}$ is distributed over Γ such that the corresponding piecewise linear curve connecting z_k approximates Γ as $N \to +\infty$. In fact, we require that

$$z_k = \gamma(t_k), \quad a = t_0 < t_1 < ... < t_N = b.$$

The limit is such that $N \to +\infty$, $\max_k |t_{k+1} - t_k| \to 0$.

We shall consider these integrals for continuous functions only (at least, continuous in a neighborhood of Γ), and for piecewise differentiable γ (i.e., for γ with bounded but not necessary continuous derivative $\gamma'(t)$). In this case, the proof of existence of the limit and its independence from the choice of $\{z_k\}$ is the same as for the standard Riemann sums in real analysis.

Calculation of the integral using the parametrization

Theorem 3.3 *Let $\gamma(t)$ be differentiable, then*

$$\int_\Gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Proof. Note that

$$z_{k+1} - z_k = \gamma(t_{k+1}) - \gamma(t_k) = \gamma'(t_k)(t_{k+1} - t_k) + o(t_{k+1} - t_k).$$

Example 3.4 *Let Γ be a curve being the image of $[a,b]$ for the mapping $\gamma(t) = Re^{it}$, where $R > 0$ is given. Let $[a,b] = [0,\pi]$, $f(z) = z$. Then*

$$\int_\Gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_0^\pi \gamma(t)\gamma'(t)dt = \int_0^\pi Re^{it}iRe^{it}dt = iR^2 \int_0^\pi e^{2it}dt$$
$$= iR^2\frac{1}{2i}e^{2ti}\Big|_0^\pi = R^2\frac{1}{2}e^{2ti}\Big|_0^\pi = 0.$$

Note that this integral does not depend on R.

Example 3.5 *In the previous example, take $[a,b]=[0,\pi/2]$. Then*

$$\int_\Gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_0^{\pi/2}\gamma(t)\gamma'(t)dt = \int_0^{\pi/2} Re^{it}iRe^{it}dt = iR^2\int_0^{\pi/2} e^{2it}dt$$
$$= iR^2\frac{1}{2i}e^{2ti}\Big|_0^{\pi/2} = R^2\frac{1}{2}e^{2ti}\Big|_0^{\pi/2} = R^2\frac{1}{2}(-1-1) = -R^2.$$

Example 3.6 *Consider previous examples with $[a,b]=[0,2\pi]$, $f(z)=1/z$. Then*

$$\int_\Gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_0^{2\pi}\gamma(t)^{-1}\gamma'(t)dt = \int_0^{2\pi} R^{-1}e^{-it}iRe^{it}dt = i\int_0^{2\pi} dt$$
$$= it|_0^{2\pi} = 2\pi i.$$

This integral does not depend on R.

Example 3.7 *Let Γ be a curve being the image of $[a,b]$ for the mapping $\gamma(t)=a+Re^{it}$, where $R>0$ is given. Let $[a,b]=[0,2\pi]$, $f(z)=1/(z-a)$. Then*

$$\int_\Gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_0^{2\pi}\gamma(t)^{-1}\gamma'(t)dt = \int_0^{2\pi} R^{-1}e^{-it}iRe^{it}dt = i\int_0^{2\pi} dt$$
$$= it|_0^{2\pi} = 2\pi i.$$

Note that this integral does not depend on a and R.

Example 3.8 *Let Γ be a curve being the image of $[a,b]$ for the mapping $\gamma(t)=Re^{it}$, where $R>0$ is given. Let $[a,b]=[0,p]$, $p>0$ $f(z)=z^{-2}$. Then*

$$\int_\Gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_0^{p}\gamma(t)^{-2}\gamma'(t)dt = \int_0^{p} R^{-2}e^{-2it}iRe^{it}dt = iR^{-1}\int_0^{p} e^{-it}dt$$
$$= -R^{-1}e^{-it}|_0^p = R^{-1}[1-e^{-ip}].$$

Note that this integral does depend on R. In addition, it follows that

$$\int_\Gamma f(z)dz = -\left(\frac{1}{z_p}-\frac{1}{z_0}\right),$$

where $z_0=\gamma(0)$, $z_p=\gamma(p)$.

The results in these examples are very significant, we shall return to them later.

Definition 3.9 *We say that a closed curve is tracing out in a positive direction, if it is anti-clockwise.*

3.3 Properties of integrals

Lemma 3.10 *(a) Let Γ be a curve given parametrically as $\gamma : [a,b] \to \mathbf{C}$. Let Γ_1 be a curve given parametrically as $\gamma : [a,c] \to \mathbf{C}$, $a < c < b$. Led Γ_2 be a curve given parametrically as $\gamma : [c,b] \to \mathbf{C}$. Then*

$$\int_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz.$$

(b) Let f, g be two functions, $\alpha \in \mathbf{C}$. Then

$$\int_\Gamma (f(z) + g(z))dz = \int_\Gamma f(z)dz + \int_\Gamma g(z)dz, \quad \int_\Gamma \alpha f(z)dz = \alpha \int_\Gamma f(z)dz.$$

(c) Let Γ_- be a curve given parametrically as $\gamma_- : [a,b] \to \mathbf{C}$, where $\gamma_-(t) = \gamma(-t+a+b))$. Then

$$\int_{\Gamma_-} f(z)dz = -\int_\Gamma f(z)dz.$$

Lemma 3.11 *Let Γ_i be a curve given parametrically as $\gamma_i : [a_i, b_i] \to \mathbf{C}$, $i = 1, 2$, where $\gamma_1(t) = \gamma_2(h(t)))$, $h : [a_1, b_1] \to [a_2, b_2]$ is a continuous strictly monotonic bijection. Then*

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz.$$

In other words, the integral does not depend on the parametrization (in class of the parametrizations that produce the same curve considered as a set, when it is taken into account the direction and how many time a point is passed). Examples 3.4-3.8 confirm that.

Lemma 3.12 *Let Γ be a curve given parametrically as $\gamma : [a,b] \to \mathbf{C}$, where $[a,b]$ is a finite interval. Then*

$$\left| \int_\Gamma f(z)dz \right| \le \max |f(z)| L,$$

where L is the length of the curve (it is taken into account how many time a point is passed).

Proof Note that

$$\left| \int_\Gamma f(z)dz \right| \sim \left| \sum_k f(z_k)(z_{k+1} - z_k) \right| \le |\sum_k f(z_k)||(z_{k+1} - z_k)|$$
$$\le |\max_k f(z_k)| \sum_k |(z_{k+1} - z_k)| \sim |\max_k f(z_k)| L.$$

Definition 3.13 *A curve Γ given parametrically as $\gamma : [a,b] \to \mathbf{C}$, is said to be C^k-smooth, if the derivatives $d^m\gamma/dt^m$ exist for $m = 0, 1, ..k$, and they are continuous. A curve is said to be piecewise C^k-smooth, if it can be represented as the union of C^k-smooth curves (as in Lemma 3.10).*

Starting from now and up to the end of these lecture notes, we consider only piecewise C^1-smooth curves.

Lemma 3.14 *Assume that curves Γ and Γ_ε are given parametrically as $\gamma : [a,b] \to \mathbf{C}$ and $\gamma_\varepsilon : [a,b] \to \mathbf{C}$ such that*

$$\max_{t\in[a,b]}(|\gamma_\varepsilon(t)-\gamma(t)|+|\gamma_\varepsilon'(t)-\gamma'(t)|)\le\varepsilon.$$

Let f be a continuous function. Then

$$\left|\int_\Gamma f(z)dz-\int_{\Gamma_\varepsilon} f(z)dz\right|\le C\max|f(z)|\varepsilon,$$

where $C>0$ does not depend on ε.

It follows from approximation results for real functions that one can approximate a integral along a piecewise C^1-continuous curve $\int_\Gamma f(z)dz$ by $\int_{\Gamma_\varepsilon} f(z)dz$ for some C^2-smooth curves Γ_ε.

On interchange of summation and integration

Lemma 3.15 *Let Γ be a path of finite length L, and let U, u_k be continuous functions on L. Assume that $\sum_{k=0}^n u_k(z)\to U(z)$ as $n\to+\infty$, and $|u_k(z)|\le M_k$ for all $z\in\Gamma$, and $\sum_{k=1}^\infty M_k<+\infty$. Then*

$$\int_\Gamma\sum_{k=0}^\infty u_k(z)dz=\sum_{k=0}^\infty\int_\Gamma u_k(z)dz.$$

Proof can be found in Priestley (2006), Chapter 14.

3.4 Integral for the case when $f(z)$ has an antiderivative

Theorem 3.16 *(The Fundamental Theorem of Calculus). Let $D\subset\mathbf{C}$ be an open set. Let Γ be a curve given parametrically by $\gamma : [a,b]\to D$. Let $f : D\to\mathbf{R}$ be a function such that there exist a holomorphic function $F(z) : D\to\mathbf{C}$ such that $F'(z)\equiv f(z)$. Then*

$$\int_\Gamma f(z)dz=F(\gamma(b))-F(\gamma(a)).$$

In particular, if Γ is a closed curve, then $\int_\Gamma f(z)dz=0$.

By The Fundamental Theorem of Calculus for real variables, we have that

$$F(\gamma(b))-F(\gamma(a))=\int_a^b\frac{d}{dt}(F(\gamma(t)))dt=\int_a^b\frac{dF}{dt}(\gamma(t))\gamma'(t)dt=\int_a^b f(\gamma(t))\gamma'(t)dt.$$

Hence

$$F(\gamma(b))-F(\gamma(a))=\int_\Gamma f(z)dz.$$

Problem 3.17 *Verify that the last theorem does not contradict to Examples 3.4-3.8.*

Corollary 3.18 *(Cauchy Theorem: the case when atiderivative exists). Let $D \subset \mathbf{C}$ be an open set. Let $f : D \to \mathbf{R}$ be a function such that there exist a holomorphic function $F(z) : D \to \mathbf{C}$ such that $F'(z) \equiv f(z)$. Let Γ_k be curves given parametrically by $\gamma_k : [a, b] \to D$ for $k = 0, 1$, such that*

$$\gamma_0(a) = \gamma_1(a), \quad \gamma_0(b) = \gamma_1(b).$$

Then

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz.$$

Proof. It suffices to see that

$$\int_{\Gamma_k} f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

□

The question arises when a function has an antiderivative.

3.5 Independence from the paths for integrals

Note that any mapping given parametrically by $\hat{\gamma} : [a, b] \to D$ can be also given parametrically by $\gamma : [0, 1] \to D$ if $\hat{\gamma}(t) = \gamma((t - a)/(b - a))$.

Definition 3.19 *Let Γ_k be closed curves given parametrically by $\gamma_k : [0, 1] \to D$ for $k = 0, 1$. We say that the paths are homotopic in D if there exists a continuous function $G : [0, 1] \times [0, 1] \to D$ such that for each s $G(, s)$ is a closed path with $G(t, 0) \equiv \gamma_0(t)$, and $G(t, 1) \equiv \gamma_1(t)$.*

This is an equivalence relation, written $\Gamma_0 \sim \Gamma_1$ in D.

We do not exclude the case when $G(t, 1) \equiv z_0 \in D$. In that case, we say that Γ_0 is homotopic to 0.

Definition 3.20 *A domain G is simply-connected if every closed path in G is homotopic to 0.*

Theorem 3.21 *(Cauchy Theorem). Let $D \subset \mathbf{C}$ be an open set. Let $f : D \to \mathbf{R}$ be a holomorphic function on D. Let Γ_k be two closed homotopic curves given parametrically by $\gamma_k : [0, 1] \to D$, where γ_k are piecewise C^1-smooth functions, $k = 0, 1$. Then*

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz.$$

Proof. For simplicity, we give the proof only for the case when homotopy is defined by a function $G(t,s) : [0,1] \times [0,1] \to \Delta$ that is twice differentiable in (t,s). Let $J(s) = \int_0^1 f(G(t,s))G'_t(t,s)dt$. We have that $J(0) = \int_{\Gamma_0} f(z)dz, \quad J(1) = \int_{\Gamma_1} f(z)dz$. Note that

$$\begin{aligned}\frac{d}{ds}[f(G(t,s))G'_t(t,s)] &= f'(G(s,t))G'_s(t,s)G'_t(s,t) + f(G(s,t))G''_{st}(s,t) \\ &= \frac{d}{dt}[f(G(t,s))G'_s(t,s)].\end{aligned}$$

It follows that

$$\begin{aligned}J'(s) = \int_0^1 \frac{d}{ds}[f(G(t,s))G'_t(t,s)]dt = \int_0^1 \frac{d}{dt}[f(G(t,s))G'_s(t,s)]dt = f(G(t,s))G'_s(t,s)\Big|_0^1 \\ = f(G(1,s))G'_s(1,s) - f(G(0,s))G'_s(0,s).\end{aligned}$$

Note that $G(1,s) \equiv G(0,s)$, hence $G'_s(1,s) \equiv G'_s(0,s)$. Hence $J'(s) \equiv 0$, i.e., $J(0) = J(1)$. □

Corollary 3.22 *(Cauchy Theorem). Let $D \subset \mathbf{C}$ be an open set. Let $f : D \to \mathbf{R}$ be a holomorphic function. Let Γ_k be curves given parametrically by $\gamma_k : [a,b] \to D$ for $k = 0,1$, such that*

$$\gamma_0(a) = \gamma_1(a), \quad \gamma_0(b) = \gamma_1(b),$$

and the closed curve $\Gamma_\cup$ is homotopic to 0, where $\Gamma_\cup$ is obtained as the union of Γ_0 with the curve with papametrization $\gamma_1^-(t) = \gamma_1(b+a-t)$. Then

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz.$$

Proof is straightforward.

Corollary 3.23 *Let $D \subset \mathbf{C}$ be an open set. Let $f : D \to \mathbf{R}$ be a holomorphic function. Let Γ be a closed curve homotopic to 0. Then*

$$\int_\Gamma f(z)dz = 0.$$

3.6 Cauchy Theorem: representation of holomorphic functions

Lemma 3.24 *Let $D \subset \mathbf{C}$ be an open simply connected set. Let $f : D \to \mathbf{R}$ be a holomorphic function, and let $a \in D$. Let Γ_R be the circle curves given parametrically by $\gamma : [0,2\pi] \to \mathbf{C}$ with $\gamma_R(t) = a + Re^{it}$ for some $R > 0$ such that $\{z : \ |z-a| \le R\} \subset D$ (in particular, $\gamma_R(t) \in D$ for all t). Then*

$$f(a) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{f(z)}{z-a}dz.$$

Proof. Consider the open domain $D_0 \triangleq D\backslash\{a\}$. Note that the function $\frac{f(z)}{z-a}$ is holomorphic in D_0, and that all curves Γ_r for small $r < R$ are mutually homotopic in D_0. Therefore,

$$J(r) \triangleq \int_{\Gamma_r} \frac{f(z)}{z-a} dz$$

does not depend on r. Further, note that

$$J(r) = \int_0^{2\pi} \frac{f(a+re^{it})}{re^{it}} rie^{it} dt = i \int_0^{2\pi} f(a+re^{it}) dt \to i \int_0^{2\pi} f(a) dt = 2\pi i f(a) \quad \text{as} \quad r \to 0.$$

We have used here the fact that the function $f|_{\Gamma_r}$ is bounded (since f is continuous and $\Gamma_r \subset D$ is a closed bounded set). We have used also The Lebesgue Dominates Convergence Theorem: if $g_k(t) \to g(t)$ for all t, and $|g_k(t)| \leq \text{const}$, then $\int_a^b g_k(t)dt \to \int_a^b g(t)dt$. Then the proof follows. □

Problem 3.25 *Constract explicitely a homotopy between Γ_r with different r in the previous proof (i.e., find explicitly the function G described in Definition 3.19).*

Theorem 3.26 *(Cauchy Formula for representation of holomorphic functions via the value on boundary). Let $D \subset \mathbf{C}$ be an open simply connected set. Let $f : D \to \mathbf{R}$ be a holomorphic function, and let $a \in D$. Let Γ be a closed curve in the domain D. Let $D_0 \triangleq D\backslash\{a\}$. Let Γ_R be the circle curves given parametrically by $\gamma_R : [0, 2\pi] \to \mathbf{C}$ with $\gamma_R(t) = a + Re^{it}$ for some $R > 0$ such that $\gamma_R(t) \in D$ for all t. Assume that Γ_R is homotopic to the curve Γ in the domain D_0. Then*

$$f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} dz.$$

Proof is straightforward; it follows immediately from Lemma 3.24 and Theorem 3.21.

Remark 3.27 *The interior part of a domain surrounded by Γ can be considered as a domain with boundary Γ, the last corollary says that the value inside domain of a holomorphic function is uniquely defined by its values on the domain boundary.*

Theorem 3.28 *(Liouville's Theorem). If a function f is holomorphic and bounded in the complex plain $\mathbf{C}$, then it is constant.*

Proof. Suppose $|f| \leq M$, $M > 0$. Let $R > 0$ be such that $|z - b| > R/2$ and $|z - a| > R/2$ for all $z = Re^{it}$, $t \in \mathbf{R}$. Let Γ be the circle $\gamma(t) = Re^{it}$, $t \in [0, 2\pi]$. We have

$$f(a) - f(b) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} dz - \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-b} dz = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)(a-b)}{(z-a)(z-b)} dz.$$

Hence

$$|f(a) - f(b)| \leq \frac{1}{2\pi} 2\pi RM \frac{|a-b|}{R^2/4}.$$

For $R \to +\infty$, $|f(a) - f(b)| \to 0$.

Theorem 3.29 *Under the asssumptions of Theorem 3.26, f has derivatives of any order $n > 0$ at a, and*

$$\frac{d^n f}{da^n}(a) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-a)^{n+1}}dz.$$

Theorem 3.30 *If a function f is holomorphic in a open domain D, then it has derivatives of any order.*

Proof will be given in the classroom.

The Fundamental Theorem of Algebra: a partial proof

Theorem *Any polynomial of order $n \in \{1, 2, 3...\}$*

$$P(z) = z^n + c_{n-1}z^{n-1} + ... + c_1 z + c_0$$

has at least one root in $\mathbf{C}$.

Proof. Suppose that the theorem statement is not true, i.e, $P(z) \neq 0$. We have that $|P(z)| \to +\infty$ as $|z| \to +\infty$. There exist $R > 0$ such that $|P(z)| > 1$ if $|z| > R$. Let $p(z) \triangleq 1/P(Z)$. This function is bounded and holomorphic, i.e. it is constant. □

3.7 Taylor series

Definition 3.31 *A function $f(z)$ is said to be analytic at a point $a \in \mathbf{C}$ if it has derivatives of all orders at this point and there exists $R = R(a) > 0$ such that*

$$f(z) = \sum_{k=0}^{\infty} c_k(z-a)^k, \quad \textit{where} \quad c_k = \frac{1}{k!}\frac{d^k f}{dz^k}(a), \tag{3.1}$$

for all $z \in D_R = \{z : \ |z-a| < R\}$, and this series absolutely converges in D_R.

Clearly, any analytic in a domain function is holomorphic. The following theorem shows if a function is holomorphic in a domain than it is analytic in the same domain.

Theorem 3.32 *Let $\hat{D} \subset \mathbf{C}$ be an open simply connected set. Let $f : \hat{D} \to \mathbf{R}$ be a holomorphic function, and let $a \in \hat{D}$. Let Γ_R be a closed curve in the domain $\hat{D}$ described as $\gamma(t) = a + Re^{it}$, where $R > 0$. Let D_R be the open disc with the boundary Γ_R. Then (3.1) holds, and this series absolutely converges in D_R.*

Proof. Let $z \in D_R$. Let r be such that $|z-a| < r < R$. For the circle D_r, we have

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(w)}{w-z}dw.$$

Let $\alpha = \alpha(w,z,a) \triangleq \frac{z-a}{w-a}$. Note that $|\alpha| = \left|\frac{z-a}{w-a}\right| < 1$, and $1-\alpha = \frac{w-z}{w-a}$. Hence

$$\frac{1}{w-z} = \frac{1}{w-a} \cdot \frac{1}{1-\alpha} = \frac{1}{w-a}(1+\alpha+\alpha^2+\alpha^3+...).$$

Hence

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(w)}{w-a}dw \sum_{k=0}^{\infty} \frac{(z-a)^k}{(w-a)^k} = \sum_{k=0}^{\infty}(z-a)^k \frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(w)}{w-a}dw \frac{1}{(w-a)^k}$$
$$= \sum_{k=0}^{\infty}(z-a)^k \frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(w)}{(w-a)^{k+1}}dw = \sum_{k=0}^{\infty}(z-a)^k \frac{1}{k!}\frac{d^k f}{dz^k}(a).$$

Uniqueness of power series representation

Theorem 3.33 *Let $\hat{D} \subset \mathbf{C}$ be an open simply connected set. Let $f : \hat{D} \to \mathbf{R}$ be a holomorphic function, and let $a \in \hat{D}$. Let Γ_R be a closed curve in the domain $\hat{D}$ described as $\gamma(t) = a + Re^{it}$, where $R > 0$. Let D_R be the open disc with the boundary Γ_R. Let*

$$f(z) = \sum_{k=0}^{\infty} c_k(z-a)^k,$$

for all $z \in D_R = \{z : \ |z-a| < R\}$, where $c_k \in \mathbf{C}$ are such that this series absolutely converges in D_R. Then $c_k = \frac{1}{k!}\frac{d^k f}{dz^k}(a)$.

Proof. We have that

$$\frac{1}{n!}\frac{d^n f}{dz^n}(a) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{f(w)}{(w-a)^{n+1}}dw = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{1}{(w-a)^{n+1}}dw \sum_{k=0}^{\infty} c_k(w-a)^k$$
$$= \sum_{k=0}^{\infty} c_k \frac{1}{2\pi i}\int_{\Gamma_R} \frac{1}{(w-a)^{n+1-k}}dw = c_n,$$

since

$$\int_{\Gamma_R} \frac{1}{(w-a)^{n+1-k}}dw = \int_0^{2\pi} R^{k-n-1}e^{-it(n+1-k)}iRe^{it}dt = i\int_0^{2\pi} R^{k-n}e^{-it(n-k)t}dt = 2\pi i\delta_{kn},$$

where we use the Kronecker symbol: $\delta_{kk} = 1$ and $\delta_{kn} = 0$ for $k \neq n$.

Corollary 3.34 *The coefficients of the power series representation for an analytic function are uniquely defined.*

Corollary 3.35 *Let Γ_R be a closed curve in the domain $\hat{D}$ described as $\gamma(t) = a + Re^{it}$, where $R > 0$. Let D_R be the open disc with the boundary Γ_R. Let $f : D_R \to \mathbf{R}$ and $g : D_R \to \mathbf{R}$ be holomorphic (i.e., analytic) functions. Let L be the line segment connecting $\alpha, \beta \in D_R$ such that $a \in L$. If $f|_L \equiv g|_L$, then $f|_{\Gamma_R} \equiv g|_{\Gamma_R}$.*

Proof follows from the fact that all the derivatives of f and g are uniquely defined by their values on L, and the values of these functions on D_R are uniquely defined by the coefficients of the corresponding power series.

Corollary 3.36 *Let Γ_R be a closed curve in the domain $\hat{D}$ described as $\gamma(t) = a + Re^{it}$, where $R > 0$. Let D_R be the open disc with the boundary Γ_R. Let $f : D_R \to \mathbf{R}$ and $g : D_R \to \mathbf{R}$ be holomorphic (i.e., analytic) functions. Let L be any one-dimensional curve segment connecting $\alpha, \beta \in D_R$ such that $a \in L$. If $f|_L \equiv g|_L$, then $f|_{\Gamma_R} \equiv g|_{\Gamma_R}$.*

Proof follows again from the fact that all the derivatives of f and g are uniquely defined by their values on L.

3.8 Zeros of holomorphic functions

The point a is said to be a zero of a function $f(z)$, if $f(a) = 0$. The zero a is said to be isolated if there exists an ε-neighborhood of a such that does not contains zeros of f except a.

Theorem 3.37 *(Identity theorem). Let D be a connected domain, and let f be holomorphic in D. Let $Z(f)$ be the set of zeros of f in D. Let $Z(f)$ has a limit point in D. Then f is identically zero in D.*

Proof of Identity Theorem. Let $a \in D$ be such that $f(a) = 0$, let $D = D_R$ be the disc of radius R with the center a such that $D_R \subset D$. We have

$$f(z) = \sum_{k \geq 0} c_k (z-a)^k, \quad z \in D_R.$$

There are two possibilities:

(i) All the coefficients $c_k = 0$; in that case, $f|_{D_R} \equiv 0$.

(ii) $\exists m > 0$: $c_k = 0$, $k < m$, $c_m \neq 0$. Set $g(z) = (z-a)^{-m} f(z) = \sum_{k \geq 0} c_k (z-a)^{k-m}$. This series converges, $g(z)$ is analytic in D_R and hence it is holomorphic and continuous in D_R. In addition, $g(a) = c_m \neq 0$, hence there exists an neighborhood of a that does not contains zeros of g except a. Hence a is an isolated zero of f. Then the proof follows for the case when $D = D_R$. (The proof for the general case will be explained on some intuitive level; the idea is that a domain where the holomorphic function is identically zero can be extended from a small disk to the entire connected domain).

Corollary 3.38 *For two analytic functions, the points where they are equal are isolated unless these functions are identical.*

Corollary 3.39 *Supppose D is a connected domain, and a holomorphic in D function f is zero in some disc in D. Then f is zero in D.*

Example 3.40 *(i) Since $(\sin x)^2 + (\cos x)^2 = 1$ for all real x, it follows that $(\sin z)^2 + (\cos z)^2 = 1$ for all $z \in \mathbf{C}$.*

(ii) Suppose f is holomorphic in $\mathbf{C}$ and such that $f(1/n) = \sin(1/n)$, then $f \equiv \sin z$

(iii) Suppose f is holomorphic in $\mathbf{C}\backslash\{0\}$ and such that $f(z) = \sin(1/z)$ for $z = 1/\pi n$, $n = 1, 2, ...$ It does not follow that necessary $f \equiv \sin(1/z)$ for $z \neq 0$. Indeed, $f \equiv 0$ would also fit the given conditions. It does not contradict to Identity Theorem, since 0 is not in the region of holomorphy of this function.

3.9 Maximum principle

Lemma 3.41 *Let f be a holomorphic function in some domain such that $|f| \equiv \text{const}$. Then f is constant in this domain.*

Proof. Let $f(z) = u(x, y) + iv(x, y)$, u, v are real functions, $z = x + iy$. We have that $u^2 + v^2 \equiv \text{const}$, i.e., $uu'_x + vv'_x = 0$, $uu'_y + vv'_y = 0$, It follows that $uu'_x - vu'_y = 0$, i.e., $u'_y = u'_x u/v$. It follows that $(u^2 + v^2)u_x = 0$. So either $u^2 + v^2 \equiv 0$ or $u'_x \equiv v'_y \equiv 0$. Similarly, $u'_y \equiv -v'_x \equiv 0$.

Lemma 3.42 *(Local Maximum-Modulsus Principle) Let Γ_R be a closed curve in the domain $\hat{D}$ described as $\gamma(t) = a + Re^{it}$, where $R > 0$. Let D_R be the open disc with the boundary Γ_R. Let $f : \hat{D} \to \mathbf{R}$ be holomorphic (i.e., analytic). Let $a \in D_R$, and let $|f(z)| \leq |f(a)|$ for all $z \in D_R$. Then $|f(z)| \equiv |f(a)|$ for all $z \in D_R$.*

Proof. Let $r \in (0, R)$. We have

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(z)}{z - a} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t) - a} \gamma'(t) dt = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))}{re^{it}} ire^{it} dt$$
$$= \frac{1}{2\pi} \int_0^{2\pi} f(\gamma(t)) dt.$$

Hence

$$|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(\gamma(t))| dt \leq |f(a)|.$$

Hence

$$0 \leq \int_0^{2\pi} (|f(\gamma(t))| - |f(a)|) dt \leq 0.$$

Hence $|f(z)| \equiv |f(a)|$ for $z \in \Gamma_r$. Since r can be any, it follows that $|f(z)|$ is constant. Hence f is constant.

Theorem 3.43 *(Maximum-Modulsus Principle). Let D be a bounded simply connected domain. Let $f : D \to \mathbf{R}$ be holomorphic (i.e., analytic) function, such that f is continuous on the closed domain $\overline{D} = D \cup \partial D$ (where ∂D is the boundary of D). Then $|f|$ attains its maximum on ∂D. If $|f|$ attains its supremum on D, then f is constant on $\overline{D}$.*

Proof. $|f|$ attains its maximum M on $\overline{D}$. Let it is attained on $a \in D$, then $|f|$ is constant on some neighborhood of a, by Lemma 3.42. Hence f is constant on this neighborhood, by Lemma 3.41. Hence it is constant on D and on $\overline{D}$.

Corollary 3.44 *Under assumptions of Theorem 3.43,* $\operatorname{Re} f$ *attains its maximum on ∂D. If* $\operatorname{Re} f$ *attains its supremum on D, then* $\operatorname{Re} f$ *is constant on $\overline{D}$.*

Proof. Apply Theorem 3.43 for $e^{f(z)}$.

Chapter 4

Laurent series

We saw that the functions $1/(z-a)$ are important for analysis. In fact, they are also very important for applications. They are not continuous at a and cannot be decomposed to Taylor series in neighborhoods of a, so we need different approach for them.

4.1 Laurent series

Definition 4.1 *A function $f(z)$ defined is some neighborhood of $a \in \mathbf{C}$ (but not necessary in a) is said to be represented as a Laurent series (or Laurent expansion) if there exist $r, R \in \mathbf{R}$, $c_k \in \mathbf{C}$ such that $0 \le r < R$,*

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k, \tag{4.1}$$

for all $z \in D_R = \{z : \ r < |z-a| < R\}$, and this series absolutely converges.

Theorem 4.2 *Let $\hat{D} \subset \mathbf{C}$ be an open simply connected set, $D_0 \subset \hat{D}$ be a closed disk with radius r_0 (the case of $r_0 = 0$ is not excluded). Let $f : \hat{D} \backslash D_0 \to \mathbf{C}$ be a holomorphic function. Let Γ_R be a closed curve in the domain $\hat{D} \backslash D_0$ described as $\gamma(t) = a + Re^{it}$, where $R > 0$. Let D_R be the open disc with the boundary Γ_R, such that $D_0 \subset D_R \subset \hat{D}$. Then (4.1) holds with*

$$c_k = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{(w-a)^{k+1}} dw, \tag{4.2}$$

where Γ is any closed curve homotopic to Γ_R in $\hat{D} \backslash D_0$, and this series absolutely converges in $D_R \backslash D_0$.

Proof. Without a loss of generality, we shall assume that $a = 0$ (otherwise, we may change variables from z to $z - a$). Let $z \in D_R \backslash D_0$. Let $r > 0$ be such that $r_0 < r < R$

and $z \in D_R \backslash D_r$. We have (explanation to be given in the classroom) that

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{\Gamma_r}\frac{f(w)}{w-z}dw.$$

Let $\alpha = \alpha(w,z) \triangleq \frac{z}{w}$, $\beta = \beta(w,z) \triangleq \frac{w}{z}$. Note that $|\alpha| < 1$ for $z \in \Gamma_R$, and $|\beta| < 1$ for $z \in \Gamma_r$,

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{w}\cdot\frac{1}{1-\alpha} = \frac{1}{w}(1+\alpha+\alpha^2+\alpha^3+...), \quad z \in \Gamma_R, \\ -\frac{1}{w-z} &= \frac{1}{z}\cdot\frac{1}{1-\beta} = \frac{1}{z}(1+\beta+\beta^2+\beta^3+...), \quad z \in \Gamma_r. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(w)}{z-w}dw &= \frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(w)}{w}dw\sum_{k=0}^{\infty}\frac{z^k}{w^k} = \sum_{k=0}^{\infty}z^k\frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(w)}{w}dw\frac{1}{w^k} \\ &= \sum_{k=0}^{\infty}z^k\frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(w)}{w^{k+1}}dw. \end{aligned}$$

Similarly,

$$\begin{aligned} -\frac{1}{2\pi i}\int_{\Gamma_r}\frac{f(w)}{z-w}dw &= \frac{1}{2\pi i}\int_{\Gamma_r}\frac{f(w)}{z}dw\sum_{k=0}^{\infty}\frac{w^k}{z^k} = \sum_{k=0}^{\infty}z^{-k-1}\frac{1}{2\pi i}\int_{\Gamma_r}\frac{f(w)}{w^{-k}}dw \\ &= \sum_{m=-1}^{-\infty}z^m\frac{1}{2\pi i}\int_{\Gamma_r}\frac{f(w)}{w^{m+1}}dw. \end{aligned}$$

In addition, note that the curves Γ_r and Γ_R are homotopic in $\hat{D}\backslash D_0$.

Uniqueness of Laurent series representation

Theorem 4.3 *Let $\hat{D} \subset \mathbf{C}$ be an open simply connected set, $D_0 \subset \hat{D}$ be a closed disk with radius r_0 (case of $r_0 = 0$ is not excluded). Let $f : \hat{D}\backslash D_0 \to \mathbf{C}$ be a holomorphic function. Let Γ_R be a closed curve in the domain $\hat{D}\backslash D_0$ described as $\gamma(t) = a + Re^{it}$, where $R > 0$. Let D_R be the open disc with the boundary Γ_R, such that $D_0 \subset D_R \subset \hat{D}$. Let*

$$f(z) = \sum_{k=-\infty}^{\infty} c_k(z-a)^k,$$

for all $z \in D_R\backslash D_0$, where $c_k \in \mathbf{C}$ are such that this series absolutely converges in D_R. Then (4.2) holds for all k.

Proof. We have that

$$\begin{aligned} \frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(w)}{(w-a)^{n+1}}dw &= \frac{1}{2\pi i}\int_{\Gamma_R}\frac{1}{(w-a)^{n+1}}dw\sum_{k=-\infty}^{\infty}c_k(w-a)^k \\ &= \sum_{k=-\infty}^{\infty}c_k\frac{1}{2\pi i}\int_{\Gamma_R}\frac{1}{(w-a)^{n+1-k}}dw = c_n, \end{aligned}$$

since

$$\int_{\Gamma_R} \frac{1}{(w-a)^{n+1-k}} dw = \int_0^{2\pi} R^{-(n+1-k)} e^{-it(n+1-k)} iRe^{it} dt = i \int_0^{2\pi} R^{-n+k} e^{-it(n-k)t} dt = 2\pi i \delta_{kn},$$

where we use the Kronecker symbol: $\delta_{kk} = 1$ and $\delta_{kn} = 0$ for $k \neq n$.

Corollary 4.4 *The coefficients of the Laurent series representation are uniquely defined (given f and a).*

4.2 Examples of Laurent expansion

Remark 4.5 *Remind that*

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + ..., \quad |z| < 1. \tag{4.3}$$

It will be useful to note that

$$\frac{1}{1-z} = -\frac{z^{-1}}{1-z^{-1}} = -z^{-1}(1 + z^{-1} + z^{-2} + ...) = -(z^{-1} + z^{-2} + z^{-3} + ...), \quad |z| > 1.$$

Example 4.6 *Let $A = \{0 < |z| < 1\}$, $f(z) = 1/[z(1-z)]$. Note that f is holomorphic in A, and $f(z) = z^{-1} + (1-z)^{-1}$, hence $f(z) = \sum_{n=-1}^{\infty} z^n$ for $z \in A$.*

Example 4.7 *Let $A = \{1 < |z| < 10^6\}$, $f(z) = 1/[z(1-z)]$. Note that f is holomorphic in A, and $f(z) = z^{-1} + (1-z)^{-1} = z^{-1} - z^{-1} - z^{-2} - z^{-3} - ....,\quad z \in A$, hence $f(z) = -\sum_{n=-\infty}^{-2} z^n$ for $z \in A$.*

Example 4.8 *Let $A = \{0 < |z| < 2\}$, $f(z) = 1/[z(1-z/2)]$. Note that f is holomorphic in A, and $f(z) = z^{-1} + (1-z/2)^{-1}/2$, hence $f(z) = z^{-1} + \sum_{n=0}^{\infty} (z/2)^n/2$ for $z \in A$.*

Example 4.9 *Let $A = \{2 < |z| < 5\}$, $f(z) = 1/[z(1-z/2)]$. We have that f is holomorphic in A, and*

$$f(z) = z^{-1} + (1-z/2)^{-1}/2 = z^{-1} - [(z/2)^{-1} + [(z/2)^{-2} + (z/2)^{-3} + ...]/2,$$

hence $f(z) = -\frac{1}{2}\sum_{n=-\infty}^{-2} (z/2)^n$ for $z \in A$.

Example 4.10 *Let $A = \{0 < |z-1| < 1\}$, $f(z) = 1/[z(1-z)^2]$. We have that f is holomorphic in A, and*

$$f(z) = \frac{1}{(1-z)^2(1+(z-1))} = \frac{1}{(1-z)^2}[1 - (z-1) + (z-1)^2 + (z-1)^3 + ...],$$

so $f(z) = \sum_{n=-2}^{\infty} (-1)^n (z-1)^n$ for $z \in A$.

Example 4.11 *Let* $A = \{0 < |z| < \pi\}$*,* $f(z) = cosec\, z = 1/\sin z$*. We have that* f *is holomorphic in* A*, and*

$$f(z) = (z - z^3/3! + z^5/5! - ...)^{-1} = z^{-1}(1 - z^2/3! + h(z))^{-1},$$

where $h(z) = O(z^4)$ *(we use the conventional* O*-notation). By (4.3),*

$$z^{-1}(1 - z^2/3! + O(z^4))^{-1} = z^{-1}(1 + z^2/3! + O(z^4))$$

for small z*. Then* $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$*, where* $c_k = 0$ *for* $k < -1$*,* $c_{-1} = 1$*,* $c_1 = 1/6$*. By taking more terms in the above expansion, we could compute* c_2*,* c_3*, etc.*

4.3 Classification for poles and singularities

Definition 4.12 a *is said to be a regular point if* f *is holomorpic at* a*.* a *is said to be a singularity of* f *if* a *is a limit of regular point and* a *is not itself regular.* a *is said to be a isolated singularity of* f *if* a *is there exist* $r > 0$ *such that* f *is holomorphic in* $\{0 < |z - a| < r\}$*.*

Definition 4.13 *Let* a *be an isolated singularity, then* f *can be represented as Laurant series* $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k$*, for* $z : 0 < |z - a| < r$ *for some* $r > 0$*.*

(a) *The function* $\sum_{k=-\infty}^{-1} c_k (z-a)^k$ *is called the principal part of the Laurent expansion for* f*.*

(b) *If* $c_k = 0$ $(\forall k < 0)$*, then* a *is said to be a removable singularity.*

(c) *If there exists* $m < 0$ *such that* $c_m \neq 0$ *and* $c_n = 0$ *for all* $n < m$*, then* a *is said to be a pole of order* m *(we call it a simple pole if* $m = 1$*, a double pole if* $m = 2$*, a triple pole if* $m = 3$*, etc).*

(d) *If there exist infinitely many* $k < 0$ *such that* $c_k \neq 0$*, then* a *is said to be an essential isolated singularity.*

Example 4.14 $1/(z-1)^2$ *has double pole at* 1*.* $1/(z^2+1)$ *has simple pole at* i*.* $1/(z^4+i)$ *has four simple poles.*

Note that these definitions are meaningful since uniqueness of the coefficient for Laurant expansions.

Problem 4.15 *Describe points* a *in the previous examples for Laurant expansion about* a*.*

4.4 Cauchy's residue theorem

Lemma 4.16 *Let $\hat{D} \subset \mathbf{C}$ be an open simply connected set. Let f be a function being holomorphic in $\hat{D}$ except a, where f has a pole, and let*

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k,$$

for all $z \in \hat{D}$, where $c_k \in \mathbf{C}$ are such that this series absolutely converges. Let Γ be a closed positively oriented curve homotopic in $\hat{D}\backslash\{a\}$ to a circle Γ_r with the center at a such that $\Gamma_R \subset \hat{D}$. Then

$$\int_\Gamma f(z)dz = 2\pi i c_{-1}.$$

Proof. We have that

$$\frac{1}{2\pi i}\int_\Gamma f(w)dw = \frac{1}{2\pi i}\int_{\Gamma_R} f(w)dw = \frac{1}{2\pi i}\int_{\Gamma_R} \sum_{k=-\infty}^{\infty} c_k (w-a)^k dw = c_{-1},$$

since

$$\int_{\Gamma_R} (w-a)^k dw = \int_0^{2\pi} R^k e^{itk} iRe^{it} dt = i\int_0^{2\pi} R^{k+1} e^{it(k+1)t} dt = 2\pi i \delta_{k,-1},$$

where we use the Kronecker symbol: $\delta_{kk} = 1$ and $\delta_{kn} = 0$ for $k \neq n$.

Definition 4.17 *In (4.4), c_{-1} is said to be the residue of f at a. We denote it as $c_{-1} = \mathrm{Res}(f, a)$.*

Theorem 4.18 *Let $\hat{D} \subset \mathbf{C}$ be an open simply connected set. Let f be a function being holomorphic in $\hat{D}$ except a finite set $\{a_k\}_{k=1}^m$, where f has poles. Let Γ be a closed positively oriented curve that have the set $\{a_k\}_{k=1}^m$ inside. Then*

$$\int_\Gamma f(z)dz = 2\pi i \sum_{k=1}^m \mathrm{Res}(f, a_k).$$

Proof to be given in the classroom.

Proposition 4.19 *(i) Let $f(z) = g(z)/(z-a)$, where $g(z)$ is holomorphic at a. Then* $\mathrm{Res}(f, a) = g(a)$.
(ii) Let $f(z) = g(z)/(z-a)^2$, where $g(z)$ is holomorphic at a. Then $\mathrm{Res}(f, a) = g'(a)$.

Proof. Note that $g(z)$ has Taylor expansion $g(z) = g(a) + g'(a)(z-a) + ...$ Then the proof follows. □

Remark 4.20 *Note that Cauchy formula (Theorem 3.26) follows from Theorem 4.18 and Proposition 4.19.*

4.5 Application to real integrals

By the definition, $\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to+\infty} \int_{-R}^{R} f(x)dx$ (if the limit exists).

For $R > 0$, let A_R be the upper half of circle $|z| = R$ (i.e., it is an arc), and let $I_R \subset \mathbf{R}$ and be the interval $[-R, R]$. Let $\Gamma(R)$ be the closed curve consisting of A_R and I_R. We assume that $\Gamma(R)$ is positively oriented.

We shall use the trivial inequality $|z| - |\alpha| \leq |z + \alpha|$.

Example 4.21 *Calculate* $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Solution. Let $f(z) = 1/(z^2 + 1)$.

(A) Let us calculate first the integral $J \triangleq \int_{\Gamma(R)} \frac{dz}{1+z^2}$.

We have $z^2 + 1 = (z + i)(z - i)$. Hence $z = i$ is the only singularity point inside $\Gamma(R)$ for $R > 1$, and $J = 2\pi i \text{Res}(f, i)$. We have $f(z) = (z - i)^{-1} g(z)$, where $g(z) = (z + i)^{-1}$ is holomorphic at $z = i$, hence it has Taylor expansion $g(z) = g(i) + g'(i)(z - i) + ...$ It follows that $\text{Res}(f, i) = g(i)$, i.e., $\text{Res}(f, i) = 1/(2i)$. It follows that $J = \pi$.

(B) Note that

$$J = J_A + J_I, \quad J_A \triangleq \int_{A_R} \frac{dz}{1 + z^2}, \quad J_I \triangleq \int_{I_R} \frac{dz}{1 + z^2}.$$

Let us show that $J_A \to 0$ as $R \to +\infty$. We have that

$$J_A = \int_0^{\pi} \frac{1}{1 + R^2 e^{2it}} iRe^{it} dt.$$

Hence

$$|J_A| \leq \int_0^{\pi} \frac{R}{|1 + R^2 e^{2it}|} |ie^{it}| dt \leq \int_0^{\pi} \frac{R}{|1 + R^2 e^{2it}|} dt \leq \int_0^{\pi} \frac{R}{R^2 - 1} dt = \frac{\pi R}{R^2 - 1},$$

since $R^2 - 1 \leq |R^2 e^{2it} + 1|$. Then $J_A \to 0$.

We have that $J_A + J_I = \pi$ for any large enough R, and $J_A \to 0$ as $R \to +\infty$. Hence

$$J_I = \int_{-R}^{R} \frac{dx}{1 + x^2} \to \pi \quad \text{as} \quad R \to +\infty, \quad \text{i.e.} \quad \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi.$$

Example 4.22 *Calculate* $\int_{-\infty}^{\infty} \frac{\cos x dx}{1+x^2}$.

Solution. Let $f(z) = e^{iz}/(z^2 + 1)$.

(A) Let us calculate first the integral $J \triangleq \int_{\Gamma(R)} \frac{e^{iz} dz}{1+z^2}$.

We have $z^2 + 1 = (z + i)(z - i)$. Hence $z = i$ is the only singularity point inside $\Gamma(R)$ for $R > 1$, and $J = 2\pi i \text{Res}(f, i)$. We have $f(z) = (z - i)^{-1} g(z)$, where $g(z) = e^{iz}(z + i)^{-1}$ is holomorphic at $z = i$, hence it has Taylor expansion $g(z) = g(i) + g'(i)(z - i) + ...$ It follows that $\text{Res}(f, i) = g(i)$, i.e., $\text{Res}(f, i) = e^{i^2}/(2i) = e^{-1}/(2i)$. It follows that $J = e^{-1}\pi$.

(B) Note that

$$J = J_A + J_I, \quad J_A \triangleq \int_{A_R} \frac{e^{iz}dz}{1+z^2}, \quad J_I \triangleq \int_{I_R} \frac{e^{iz}dz}{1+z^2}.$$

Let us show that $J_A \to 0$ as $R \to +\infty$. We have that

$$J_A = \int_0^\pi \frac{e^{iRe^{it}}}{1+R^2e^{2it}} iRe^{it}dt.$$

Hence

$$|J_A| \le \int_0^\pi \frac{R|e^{iRe^{it}}|}{|1+R^2e^{2it}|}|ie^{it}|dt \le \int_0^\pi \frac{R}{|1+R^2e^{2it}|}dt \le \int_0^\pi \frac{R}{R^2-1}dt = \frac{\pi R}{R^2-1},$$

since $R^2 - 1 \le |R^2e^{2it} + 1|$ and $|e^{iRe^{it}}| \le 1$ (remind that $\operatorname{Im} z < 0$ for $z = R^{it} \in A_R$). Then $J_A \to 0$ as $R \to +\infty$.

We have that $J_A + J_I = e^{-1}\pi$ for any large enough R, and $J_A \to 0$ as $R \to +\infty$. Hence

$$\int_{-R}^R \frac{\cos x dx}{1+x^2} = \operatorname{Re} \int_{-R}^R \frac{e^{ix}dx}{1+x^2} = \operatorname{Re} J_I \to e^{-1}\pi \quad \text{as} \quad R \to +\infty, \quad \text{i.e.} \quad \int_{-\infty}^\infty \frac{\cos x dx}{1+x^2} = e^{-1}\pi.$$

□

A question: is it possible to take $\int_{\Gamma(R)} \frac{\cos z dz}{1+z^2}$ instead of $\int_{\Gamma(R)} \frac{e^{iz}dz}{1+z^2}$ in the previous solution?

Example 4.23 *Given $h > 0$, calculate $\int_{-\infty}^\infty \frac{\cos(x+h)dx}{1+x^2}$.*

Solution. Let $f(z) = e^{i(z+h)}/(z^2+1)$.

(A) Let us calculate first the integral $J \triangleq \int_{\Gamma(R)} \frac{e^{i(z+h)}dz}{1+z^2}$.

We have $z^2+1 = (z+i)(z-i)$. Hence $z = i$ is the only singularity point inside $\Gamma(R)$ for $R > 1$, and $J = 2\pi i \operatorname{Res}(f,i)$. We have $f(z) = (z-i)^{-1}g(z)$, where $g(z) = e^{i(z+h)}(z+i)^{-1}$ is holomorphic at $z = i$, hence it has Taylor expansion $g(z) = g(i) + g'(i)(z-i) + ...$ It follows that $\operatorname{Res}(f,i) = g(i)$, i.e., $\operatorname{Res}(f,i) = e^{i(i+h)}/(2i) = e^{-1+ih}/(2i)$. It follows that $J = e^{-1+ih}\pi$.

(B) Note that

$$J = J_A + J_I, \quad J_A \triangleq \int_{A_R} \frac{e^{i(z+h)}dz}{1+z^2}, \quad J_I \triangleq \int_{I_R} \frac{e^{i(z+h)}dz}{1+z^2}.$$

Let us show that $J_A \to 0$ as $R \to +\infty$. We have that

$$J_A = \int_0^\pi \frac{e^{iRe^{it}+ih}}{1+R^2e^{2it}} iRe^{it}dt.$$

Remind that $\operatorname{Im} z < 0$ for $z = R^{it} \in A_R$. Hence

$$\left|\int_{A_R} \frac{e^{i(z+h)}dz}{1+z^2}\right| \le \int_0^\pi \frac{R|e^{iRe^{it}+ih}|}{|1+R^2e^{2it}|}|ie^{it}|dt \le \int_0^\pi \frac{R}{|1+R^2e^{2it}|}dt \le \int_0^\pi \frac{R}{R^2-1}dt = \frac{\pi R}{R^2-1},$$

since $R^2-1\le|R^2e^{2it}+1|$ and $|e^{iRe^{it}+ih}|=|e^{iRe^{it}}|\le 1$. Then $J_A\to 0$ as $R\to+\infty$.

We have that $J_A+J_I=e^{-1+ih}\pi$ for any large enough R, and $J_A\to 0$ as $R\to+\infty$. Hence

$$\int_{-R}^{R}\frac{\cos(x+h)dx}{1+x^2}=\mathrm{Re}\int_{-R}^{R}\frac{e^{i(x+h)}dx}{1+x^2}=\mathrm{Re}\,J_I\to e^{-1}\pi\cos h\quad\text{as}\quad R\to+\infty,$$

$$\text{i.e.}\quad\int_{-\infty}^{\infty}\frac{\cos(x+h)dx}{1+x^2}=e^{-1}\pi\cos h.$$

□

Example 4.24 *Calculate* $\int_{-\infty}^{\infty}\frac{\sin x dx}{x(1+x^2)}$.

Solution. Let $f(z)=e^{iz}/(z(z^2+1))$. Let $0<r<R$. Let $\Gamma'=\Gamma'(R,r)$ be the positively oriented closed curve that includes the arcs A_r and A_R and the linear segments connecting them.

(A) Let us calculate first the integral $J\triangleq\int_{\Gamma'}\frac{e^{iz}dz}{x(1+z^2)}$.

We have $z^2+1=(z+i)(z-i)$. Hence $z=i$ is the only singularity point inside Γ' for $R>1$ and $r<1$, and $J=2\pi i\mathrm{Res}(f,i)$. We have $f(z)=(z-i)^{-1}g(z)$, where $g(z)=e^{iz}z^{-1}(z+i)^{-1}$ is holomorphic at $z=i$, hence it has Taylor expansion $g(z)=g(i)+g'(i)(z-i)+\ldots$. It follows that $\mathrm{Res}(f,i)=g(i)$, i.e., $\mathrm{Res}(f,i)=e^{i^2}/(i\cdot 2i)=-e^{-1}/2$. It follows that $J=-e^{-1}\pi\cdot i$.

(B) Note that

$$J=J_A+J_a+J_I,\quad J_A\triangleq\int_{A_R}\frac{e^{iz}dz}{z(1+z^2)},\quad J_a\triangleq\int_{A_r}\frac{e^{iz}dz}{z(1+z^2)},\quad J_I\triangleq\int_{I_{R,r}}\frac{e^{iz}dz}{z(1+z^2)}.$$

Let us show that $J_A\to 0$ as $R\to+\infty$. We have that

$$J_A=\int_0^{\pi}\frac{e^{iRe^{it}}}{Re^{it}(1+R^2e^{2it})}iRe^{it}dt.$$

Hence

$$|J_A|\le\int_0^{\pi}\frac{|e^{iRe^{it}}|}{R|1+R^2e^{2it}|}|Rie^{it}|dt\le\int_0^{\pi}\frac{1}{|1+R^2e^{2it}|}dt\le\int_0^{\pi}\frac{1}{1(R^2-1)}dt=\frac{\pi}{R^2-1},$$

since $R^2-1\le|R^2e^{2it}+1|$ and $|e^{iRe^{it}}|\le 1$ (remind that $\mathrm{Im}\,z<0$ for $z\in A_R$). Then $J_A\to 0$ as $R\to+\infty$.

(C) We have that

$$J_a=\int_{A_r}\frac{e^{iz}dz}{z(1+z^2)}=-\int_0^{\pi}\frac{e^{ire^{it}}}{re^{it}(1+r^2e^{2it})}ire^{it}dt\to -i\int_0^{\pi}dt=-i\pi$$

as $r\to 0$.

(D) Remind that $J = J_A + J_a + J_I = -ie^{-1}\pi$ for any large enough R and small enough $r > 0$, and $J_A \to 0$ as $R \to +\infty$. Hence

$$\int_{I(R,r)} \frac{\sin x dx}{1+x^2} = \operatorname{Im} \int_{I(R,r)} \frac{e^{ix} dx}{1+x^2} = \operatorname{Im}\left(J - J_A - J_a\right) \to -e^{-1}\pi - (-\pi) = \pi(1 - e^{-1})$$

$$\text{as} \quad R \to +\infty, \qquad \text{i.e.} \quad \int_{-\infty}^{\infty} \frac{\sin x dx}{x(1+x^2)} = (1 - e^{-1})\pi.$$

□

Chapter 5

Winding numbers

5.1 Winding number: definitions

Definition 5.1 *Let a closed curve Γ be given. The winding number of Γ about the original 0 is the net number of revolutions of the directions of z as it traces out Γ once.*

Definition 5.2 *We say that a point is tracing out a closed curve in a positive direction, if it is anti-clockwise.*

Clearly, the argument of z is increasing on $2\pi\nu(\Gamma, 0)$, if the curve is traced out in the positive direction.

Definition 5.3 *Let $p \in \mathbf{C}$ and a closed curve Γ be given. The winding number $\nu(\Gamma, p)$ of Γ about the original p is the net number of revolutions as it traces out Γ once.*

Clearly, the winding number is not changing if one moves the curve slightly. The following topological result is a very strong generalization of this fact.

Theorem 5.4 *If Γ is piecewise smooth and such that $a \notin \Gamma$, then*

$$\nu(\Gamma, a) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z-a}dz.$$

Proof. If Γ is a circle with the center at a repeated a number of times, the theorem statement can be obtained by direct calculation of the integral: if $\Gamma = a + e^{it}$, $t \in [0, 2\pi m]$, then

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{z-a}dz = \frac{1}{2\pi i}\int_0^{2\pi m} \frac{1}{e^{it}} i e^{it} dt = \frac{1}{2\pi i} 2\pi i m = m.$$

For all other cases the proof follow from the fact that the integral does not change if the curve is transformed to a homotopic curve. □

Theorem 5.5 *(Hopf's degree theorem) A closed curve can be deformed to another closed curve without crossing p iff the winding number (about p) is the same for the both curves.*

Theorem 5.6 *(Cauchy Theorem). Let $D \subset \mathbf{C}$ be an open set. Let $f : D \to \mathbf{R}$ be a holomorphic function, and let $a \in D$. Let Γ be a closed curve in the domain D such as described in Theorem 3.26. Then*

$$\nu(\Gamma, a) f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} dz.$$

Here $\nu(\Gamma, a)$ is the winding number of Γ about the point a.

5.2 Argument principle and counting of roots

Definition 5.7 *We say that a is a root of multiplicity m of a function $f(z)$ if $f(z) = (z-a)^m g(z)$, where $g(z)$ is a function such that $g(a) \neq 0$.*

Theorem 5.8 *(Argument Principle) Let f be a holomorphic function. Let Γ be a closed curve in D such as described in Theorem 3.26. Let f has exactly n roots $a_1, ..., a_n$ inside Γ (counted with their multiplicity), and let $\nu(\Gamma, a_k) = 1$ for any k. Then $\nu(f(\Gamma), 0) = n$.*

Proof. Without a loss of generality, we can assume that $f(z)$ has roots $a_1, ..., a_n$ in D, then $f(z) = g(z) \prod_{k=1}^{b} (z - a_k)$, where $g(z)$ is a holomorphic function being nonzero on D. Then

$$\frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \frac{1}{z-a_2} + ... + \frac{1}{z-a_n} + \frac{g'(z)}{g(z)}.$$

By Corollary 3.23, it follows that

$$\frac{1}{2\pi i} \int_\Gamma \frac{g'(z)}{g(z)} dz = 0.$$

By Theorem 5.4, it follows that

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{n} \nu(\Gamma, a_k) = n.$$

Let $\gamma(t) : [a, b] \to \mathbf{C}$ be a parametrization of Γ. The integral here can be rewritten as

$$\frac{1}{2\pi i} \int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) dt = \frac{1}{2\pi i} \int_{f(\Gamma)} \frac{dw}{w} = \nu(f(\Gamma), 0) = n,$$

where $f(\Gamma)$ is the curve with the parametrization $f(\gamma(t))$. This completes the proof. □

Theorem 5.9 *(Rouche's theorem). Let D be a simply connected domain, and let functions f, g be holomorphic in D. Let Γ be a closed curve in D being the image of $[0, 2\pi]$ for the mapping $\gamma(t) = Re^{it}$, where $R > 0$ is given. Let $|g(z)| < |f(z)|$ for all $z \in \Gamma$. Then f and $f + g$ have the same number of roots inside Γ (counted with their multiplicity).*

Proof. First, it can be seen that

$$\nu(f(\Gamma), 0) = \nu((f+g)(\Gamma), 0).$$

To see this, one may think about a person walking around 0 along the trail $f(\Gamma)$ with a dog on the leash that is being kept shorter than the distance between the person and 0. The man's position is $f(z)$, the dog's position is $f(z)+g(z)$, $z \in \Gamma$. Clearly, the dog on the leash will makes the same number of revolutions around 0 as the person holding the leash. By Argument Principle, the required statement follows. $\square$

5.3 The Fundamental Theorem of Algebra: the proof

Theorem *Any polynomial of order* $n \in \{1, 2, 3...\}$

$$P(z) = z^n + c_{n-1}z^{n-1} + ... + c_1 z + c_0$$

has n *roots in* $\mathbf{C}$.

Proof. We have that $P(z) = f(z) + g(z)$, where $f(z) = z^n$, $g(z) = c_{n-1}z^{n-1} + ... + c_1 z + c_0$. Let Γ be a closed curve being the image of $[0, 2\pi]$ for the mapping $\gamma(t) = Re^{it}$, where $R > 0$ is such that

$$R^n > |c_{n-1}|R^{n-1} + ... + |c_1|R + |c_0|.$$

Clearly, it holds for large enough $R > 0$, say, for

$$R > n \max_k |c_k|.$$

We have that

$$|f(Re^{it})| = |R^n e^{itn}| = R^n,$$

$$\begin{aligned} |g(Re^{it})| &= |c_{n-1}(Re^{it})^{n-1} + ... + c_1 Re^{it} + c_0| \\ &\le |c_{n-1}|R^{n-1} + ... + |c_1|R + |c_0|. \end{aligned}$$

It follows that $|g(z)| < |f(z)|$ for all $z \in \Gamma$. By Rouchet's Theorem, it follows that f and $P = f + g$ have the same number of roots inside Γ (counted with their multiplicity). Remind that $f(z) = z^n$ has n zero roots. Then the proof follows. $\square$

Chapter 6

Transforms for representation of processes in frequency domain

A transform, in general, is a formula that converts one function into another function by some rule. (For example, the derivative is a kind of transform in that $f'(t)$ transforms a function, $f(x)$, into its derivative). Transforms are in fact mappings defined on classes of functions. We shall consider four important transforms that are being used widely for so-called *spectral* representation of time depending processes, or for representation of the processes in so-called *frequency domain*. In this form, a function of time is represented as a summa of oscillating processes. For instance, let $\omega \in \mathbf{R}$ be given. Then the processes $f_0(t) = \cos(\omega t)$, $f_1(t) = \sin(\omega t)$, and $f_0(t) = \exp(i\omega t)$, have the same frequency ω; they spectrum is the singleton $\{\omega\}$. If we observe a process $f(t)$ and found from measurements that $f(t) = 5\sin(\omega_1 t) - 2\cos(\omega_2 t)$ for some $\omega_k \in \mathbf{R}$, then we may say that the process $f(t)$ has spectrum $\{\omega_1, \omega_3\}$. This kind of analysis in one of the basic tools in mathematics, engineering, physics, system theory.

Define class $M(r)$ of all functions $f(\cdot) : [0, +\infty) \to \mathbf{C}$ such that there exists a constant $C > o$ such that

$$|f(t)| \leq Ce^{rt}, \quad \forall t > 0.$$

Let $I \subset \mathbf{R}$, $p \geq 1$. We denote $\mathcal{L}_p(I, \mathbf{R})$ the class of all functions $f : I \to \mathbf{R}$ such that $\int_I |f(t)|^p dt < +\infty$. Similarly, we denote $\mathcal{L}_p(I, \mathbf{C})$ the class of all functions $f : I \to \mathbf{C}$ such that $\int_I |f(t)|^p dt < +\infty$. We denote by $L_p(I, \mathbf{R})$ the class of classes of equivalency from $\mathcal{L}_p(I, \mathbf{C})$. In other words, if $\operatorname{mes}(f_1 \neq f_2) = 0$, then $f_1 = f_2$, meaning that they represents the same element of $L_p(I, \mathbf{C})$, i.e., they are in the same class of equivalency.

Sometime we denote both these classes as $\mathcal{L}^p(I)$ and $L_p(I)$.

6.1 Laplace Transform

Let $f(\cdot) \in M(r)$. Then the Laplace transform $F = \mathcal{L}f$ is

$$F(s) = \int_0^{+\infty} e^{-st} f(t) dt,$$

where $s \in \mathbf{C}$ is such that $\operatorname{Re} s > r$.

Proposition 6.1 *In the definition above, the integral exists.*

Theorem 6.2 *A function from $M(r)$ is uniquely defined by its Laplace transform. (i.e. if two functions have the same Laplace transform then they are same).*

Theorem 6.3 *Let $f(\cdot) \in M(r)$. Then $F(s)$ is holomorphic in$\{z : \operatorname{Re} z > r\}$.*

It follows that Laplace transform is uniquely defined by its values for real s only.

Clearly, the Laplace transform is a linear transform. Thus the transform may be split up, if a function is defined over a split domain.

Since the transform maps a function $f(t)$ into some function $F(s)$, it is reasonable to ask if there is an inverse function $\mathcal{L}^{-1}$ that takes $F(s)$ back to $f(t)$. In many cases the answer is yes. There are tables of such inverses and partial fractions are often used to break up rational functions.

Some important transforms:

1. For $f(t) \equiv c$, where c is a constant, the Laplace transform is c/s; $\operatorname{Re} s > 0$.
2. For $f(t) \equiv e^{at}$, where $a \in \mathbf{C}$ is a constant, the Laplace transform is $\frac{1}{s-a}$; $\operatorname{Re} s > \operatorname{Re} a$.
3. For $f(t) \equiv \sin at$, where $a \in \mathbf{R}$ is a constant, the Laplace transform is $\frac{a}{s^2+a^2}$; $\operatorname{Re} s > 0$.
4. For $f(t) \equiv \cos at$, where $a \in \mathbf{R}$ is a constant, the Laplace transform is $\frac{s}{s^2+a^2}$; $\operatorname{Re} s > 0$.
5. If $f(t)$ has the Laplace transform $F(s)$, then $e^{zt} f(t)$ has the Laplace transform $F(s-z)$.
6. For $f(t) \equiv e^{zt} \sin at$, where $a \in \mathbf{R}$, $z \in \mathbf{C}$, the Laplace transform is $\frac{a}{(s-z)^2+a^2}$; $\operatorname{Re} s > \operatorname{Re} z$.
7. For $f(t) \equiv e^{zt} \cos at$, where $a \in \mathbf{R}$, $z \in \mathbf{C}$, the Laplace transform is $\frac{s-z}{(s-z)^2+a^2}$; $\operatorname{Re} s > \operatorname{Re} z$.
8. For $f(t) \equiv t^n e^{at}$, where $a \in \mathbf{R}$, $n \in \mathbf{N}$, the Laplace transform is $\frac{n!}{(s-a)^{n+1}}$; $\operatorname{Re} s > a$.

6.1.1 Laplace transform and differentiation

Denote $\mathcal{M} \triangleq \cup_{r \in R} M(r)$.

Theorem 6.4 *Let $f(\cdot)$ and $\frac{df}{dt}(\cdot)$ belongs to $\mathcal{M}$. Then the Laplace transform for $\frac{df}{dt}(\cdot)$ is $sF(s) - f(0)$, where $F(s)$ is the Laplace transform for f.*

Proof. We have that $f(\cdot)$ and $\frac{df}{dt}(\cdot)$ belongs to $\mathcal{M}(r)$ for some r. Let $s \in \mathbf{C}$ be such that $\operatorname{Re} s > r$. Then

$$\int_0^{+\infty} e^{-st}\frac{df}{dt}(t) = e^{-st}f(t)|_0^{+\infty} - \int_0^{+\infty}(-s)e^{-st}f(t)dt = sF(s) - f(0).$$

□

Corollary 6.5 *Let $f(\cdot)$ and $\int_0^t f(s)ds$ belongs to $\mathcal{M}$. Then the Laplace transform for $\int_0^t f(s)ds$ is $\frac{F(s)}{s}$, where $F(s)$ is the Laplace transform for f.*

Proof. The Laplace transform for f is $sG(s)$, where $G(s)$ is the Laplace transform for $g(t) \triangleq \int_0^t f(s)ds$. □

Example of application

Consider a scalar ODE (ordinary differential equation)

$$\begin{cases} y'(t) = ay(t) + f(t), \\ y(0) = x. \end{cases}$$

Let $Y(s)$, $F(s)$ be the Laplace transforms for y, f correspondingly. We have $sY(s) - x = aY(s) + F(s)$, i.e.

$$Y(s) = \frac{x}{s-a} + \frac{F(s)}{s-a}$$

Thus, $y(t)$ can be found as inverse transform of $Y(s)$, or as $e^{at}x$ plus inverse transform of $\frac{F(s)}{s-a}$.

6.1.2 Convolution and the Laplace transform

Convolution of functions $f(t) : [0, +\infty) \to \mathbf{C}$ and $g(t) : [0, +\infty) \to \mathbf{C}$ is a function $f * g : [0, +\infty) \to \mathbf{R}$ defined as

$$(f * g)(t) \triangleq \int_0^t f(\tau)g(t-\tau)d\tau.$$

Theorem 6.6 *(Convolution Theorem) Let $f \in \mathcal{M}$, $g \in \mathcal{M}$, then the Laplace transform of the convolution of $f * g$ is $F(s)G(s)$, where $F(s)$ and $G(s)$ are the Laplace transforms for f and g correspondingly.*

Proof. Let $f(\cdot), g(\cdot) \in M(r)$ for some $r \in \mathbf{R}$. We have for $s \in \mathbf{C}$ such that $\operatorname{Re} s > r$

$$\int_0^{+\infty} e^{-st}dt \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^{+\infty} d\tau f(\tau) \int_\tau^{+\infty} e^{-st}g(t-\tau)dt$$
$$= \int_0^{+\infty} d\tau f(\tau) \int_0^{+\infty} e^{-s(r+\tau)}g(r)dr = \int_0^{+\infty} e^{-s\tau}f(\tau)d\tau \int_0^{+\infty} e^{-sr}g(r)dr = F(s)G(s).$$

Application for ODEs

For $y(0) = 0$, we have

$$y(t) = \exp(at) * f(t), \quad Y(s) = F(s)/(s-a). \tag{6.1}$$

Application for inverse transform

Sometimes convolution can help to find inverse transform. Let us find the inverse transform of a fraction $\frac{1}{s^2+3s-10}$. We can inverse it using partial fractions:

$$\frac{1}{s^2+3s-10} = -\frac{1}{7(s+5)} + \frac{1}{7(s-2)}.$$

Instead, we can use Convolution Theorem:

$$\frac{1}{s^2+3s-10} = \frac{1}{s+5} \cdot \frac{1}{s-2}.$$

The inverse of the Laplace transform is the convolution of e^{-5t} and e^{2t} and can be calculated is

$$e^{-5t} * e^{2t} = \int_0^t e^{-5\tau} e^{2(t-\tau)} d\tau = \frac{1}{7}(e^{2t} - e^{-5t}).$$

6.1.3 Heaviside step function and shift

Heaviside function

$$H(t) \triangleq \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

A piecewise constant function can be expressed via combination of Heaviside functions with shifts such as $H(t-a) - H(t-b)$.

We shall call $\hat{f}(t) \triangleq H(t-a)f(t-a)$ a time-delayed function; its graph is same as that for $f(t)$ but shifted to the right by a and "turned off" for all $t < a$.

Proposition 6.7 *If $f \in \mathcal{M}$ then $H(t-a)f(t-a) \in \mathcal{M}$.*

Proposition 6.8 *The Laplace transform for $H(t-a)$ is e^{-as}/s.*

Lemma 6.9 *Let $f \in \mathcal{M}$ and the Laplace for $f(t)$ is $F(s)$. Then the Laplace transform for $H(t-a)f(t-a)$ is $e^{-as}F(s)$.*

This lemma helps to find the Laplace transforms for shifted functions, but it helps also find the inverse for Laplace transforms with exponents.

Derivative of Heaviside function

Let $a > 0$. Consider a mapping $\delta(t-a) : C(0,+\infty) \to \mathbf{R}$ such that

$$\langle \delta(t-a), f(t) \rangle \triangleq \lim_{\varepsilon \to 0} \int_0^{+\infty} \delta_\varepsilon(t-a) f(t) dt = f(a) \qquad \forall f(\cdot) \in C(0,+\infty),$$

where

$$\delta_\varepsilon(t-a) = \begin{cases} 0, & |t-a| > \varepsilon \\ \frac{1}{2\varepsilon}, & |t-a| \le \varepsilon. \end{cases}$$

$\delta(t)$ is the so-called *delta-function.* The limit here is denoted usually as $\int_0^{+\infty} \delta(t-a) f(t) dt$. [1] Apply formally the Laplace transform:

$$\int_0^{+\infty} \delta(t-a) e^{-st} dt = e^{-as}.$$

The Laplace transform for $H(t-a)$ is e^{-as}/s. Thus, the Laplace transform for the delta-function is the same as the Laplace transform for the "derivative" of Heaviside function; this fact can be presented formally as $\delta(t-a) = \frac{\partial H(t-a)}{\partial t}$: it is a so-called *generalized derivative.*

Corollary 6.10 *We have that inverse of Laplace transorm for e^{-as} gives delta-function $\delta(t-s)$, which is not a function (it is a "generalized function"). This means that the inverse of Laplace transform may be not defined in the class of functions even for holomorphic functions $F(s)$.*

Applications to control and system theory

In many cases, continuous time dynamic systems are described by ODEs. Consider a simple exam:

$$\frac{dx(t)}{dt} = ax(t) + f(t),$$
$$x(0) = 0$$

In control and system theory and its applications in engineering, physics, in signal processes), the process $f(t)$ (or $F(s) = \mathcal{L}f$) is interpreted as an input of a *linear continuous time time system* with *transfer function* $\chi(s) = \frac{1}{s-a}$, i.e., $X(s) = \chi(s)F(s)$ for $X = \mathcal{L}x$. The solution process $x(t) = e^{at} * f(t) = \int_0^t e^{a(t-s)} f(s) ds$ (or $X(p)$) is interpreted as an output of this system. The same model is used for more general dynamic systems. The transfer function describes completely the properties of input-output system. It is why the methods of complex analysis are very common in system theory. For instance, the

[1] Hint: remember that $\delta(t)$ is <u>not a function of t</u>, and the integral of $\delta(t-a)f(t)$ is not an integral at all, it is just a symbol!

system from our example is "stable"[2] iff $\operatorname{Re} a < 0$. In a case of more general χ, the system is stable iff the transfer function does not have singularities in $\{z : \operatorname{Re} \geq 0\}$.

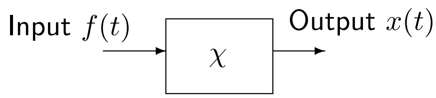

Figure 6.1: The block diagram for the system $dx(t)/dt = ax(t) + f(t)$.

Note that the Laplace transform is targeting process evolving in time on $(0, +\infty)$. For processes defined on $(-\infty, +\infty)$, we use *Fourier transform.*

6.2 Fourier Transforms

Let $\mathcal{L}_p(\mathbf{R}) = \mathcal{L}_p(\mathbf{R}, \mathbf{C})$.

For $f \in L_2(\mathbf{R})$, the Fourier transform $\hat{f} = \mathcal{F}f$ is[3]

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\omega t} f(t) dt,$$

where $\omega \in \mathbf{R}$.

Proposition 6.11 *If $f(\cdot) \in \mathcal{L}_1(\mathbf{R})$, then the integral exists for all $\omega \in \mathbf{R}$. For $f(\cdot) \in \mathcal{L}_2(\mathbf{R})$, the integral exists as an element of $L_2(\mathbf{R})$ (i.e., not necessary for all ω).*

Clearly, the Fourier transform is a linear transform.

Theorem 6.12 *(Plancherel's-Parseval's Theorem) Let $\hat{f} = \mathcal{F}f$, $\hat{g} = \mathcal{F}g$, where $f, g \in \mathcal{L}_2(\mathbf{R})$. Then*

$$\int_{\mathbf{R}} \overline{f}(t) g(t) dt = \int_{\mathbf{R}} \overline{\hat{f}(\omega)} \hat{g}(\omega) d\omega.$$

Theorem 6.13 *The mapping $\mathcal{F} : L_2(\mathbf{R}) \to L_2(\mathbf{R})$ is a bijection.*

By this theorem, there exists inverse mapping $\mathcal{F}^{-1} : L_2(\mathbf{R}) \to L_2(\mathbf{R})$ that takes $\hat{f}$ back to f. There are tables for Fourier transforms and their inverses.

Theorem 6.14 *The inverse of a Fourier transform $f = \mathcal{F}^{-1}\hat{f}$ exists for $\hat{f} \in L_2(\mathbf{R})$*

$$(\mathcal{F}^{-1}\hat{f})(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{i\omega t} \hat{f}(\omega) d\omega.$$

[2] Stability is a very important concept in theory of dynamic systems. One of many possible definitions is that a system is stable if any bounded input produces a bounded output on infinite horizon.

[3] In literature, the multiplier $\frac{1}{\sqrt{2\pi}}$ is being replaced sometimes by a different one; sometimes (but rarely enough) $e^{-i\omega t}$ is being replaced by $e^{i\omega t}$.

In literature, $\hat{f}$ is said to be a representation of f in the frequency domain; ω is frequency.

Remark 6.15 *Let f be such that $f(t) = 0$ for $t < 0$ and that Lapalce transform $F(s) = \mathcal{L}f$ is defined for all $s = i\omega$, where $\omega \in \mathbf{R}$. In that case, $\hat{f}(\omega) = (2\pi)^{-1/2}F(i\omega)$. In other words, the trace of the Laplace transform on the imaginary axe (i.e., for $s = i\omega$, $\omega \in \mathbf{R}$) is a Fourier transform (assuming that the function $f(t)$ is extended as zero on $(-\infty, 0)$.*

Because of this connection between Laplace and Fourier transforms, $F = \mathcal{L}f$ is also said to be a representation of f in the frequency domain; for $F(s)$, $\operatorname{Im} s$ is the frequency.

6.2.1 Fourier transform and differentiation

Theorem 6.16 *Let $f(\cdot)$ and $\frac{df}{dt}(\cdot)$ belongs to $L_2(\mathbf{R})$. Then the Laplace transform for $\frac{df}{dt}(\cdot)$ is $i\omega f(i\omega)$, where $\hat{f}(s)$ is the Laplace transform for f.*

Proof. We have that

$$\int_{\mathbf{R}} e^{-i\omega t}\frac{df}{dt}(t) = e^{-i\omega t}f(t)|_{-\infty}^{+\infty} - \int_{\mathbf{R}}(-i\omega t)e^{-i\omega t}f(t)dt = i\omega F(i\omega).$$

□

6.2.2 Convolution and the Fourier transform

Convolution of functions $f(t) : \mathbf{R} \to \mathbf{C}$ and $g(t) : \mathbf{R} \to \mathbf{C}$ is a function $f * g : [0, +\infty) \to \mathbf{R}$ defined as

$$(f * g)(t) \triangleq \int_{\mathbf{R}} f(\tau)g(t-\tau)d\tau.$$

Theorem 6.17 *(Convolution Theorem) Let $f, g \in \mathcal{L}_2(\mathbf{R})$, then the Fourier transform of the convolution of $f * g$ is $\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$, where $\hat{f}(\omega)$ and $\hat{g}(\omega)$ are the Fourier transforms for f and g correspondingly.*

Proof. We have for $s \in \mathbf{C}$ such that $\operatorname{Re} s > r$

$$\begin{aligned}&\int_{\mathbf{R}} e^{-i\omega t}dt\int_{\mathbf{R}} f(\tau)g(t-\tau)d\tau = \int_{\mathbf{R}} d\tau f(\tau)\int_{\mathbf{R}} e^{-i\omega t}g(t-\tau)dt\\ &= \int_{\mathbf{R}} d\tau f(\tau)\int_{\mathbf{R}} e^{-i\omega(r+\tau)}g(r)dr = \int_{\mathbf{R}} e^{-i\omega\tau}f(\tau)d\tau\int_{\mathbf{R}} e^{-i\omega r}g(r)dr = 2\pi\hat{f}(\omega)\hat{g}(\omega).\end{aligned}$$

Applications for dynamic systems: energy equality

Consider a dynamic system with transfer function $\chi(s)$. Let $f(t)$, $t > 0$, be the input process, and $x(t)$ be the output process. For $X = \mathcal{L}x$, $\hat{x} = \mathcal{F}x$, $F = \mathcal{L}f$,

$$\int_0^{+\infty}|x(t)|^2dt = \int_{-\infty}^{+\infty}|\hat{x}(\omega)|^2d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(i\omega)|^2d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|\chi(i\omega)|^2|F(i\omega)|^2d\omega.$$

For instance, let $a < 0$, then this result can be applied to for the system

$$\frac{dx(t)}{dt} = ax(t) + f(t), \quad t > 0,$$
$$x(0) = 0$$

with $\chi(s) = 1/(s-a)$, $|\chi(i\omega)|^2 = 1/(\omega^2 + a^2)$.

6.3 Fourier Series

Let l_p denotes the set of al sequences $\{c_k\}_{k=-\infty}^{\infty} \subset \mathbf{C}$ such that $\sum_{k=-\infty}^{\infty} |c_k|^p < +\infty$.

Let $\mathcal{L}_p(-\pi,\pi) = \mathcal{L}_p([-\pi,\pi],\mathbf{C})$.

Fourier series is representation of a function $f : [-\pi,\pi] \to \mathbf{C}$

$$f(t) = \sum_{k=-\infty}^{+\infty} c_k e^{ikt},$$

where $c_k \in \mathbf{C}$ are said to be the Fourier coefficients. [4]

Proposition 6.18 *(i) If $\{c_k\} \in l_1$, then the series converges.*

(ii) If $\{c_k\} \in l_2$, then the series converges in the space $L_2(-\pi,\pi)$[5] (i.e., not necessary for all t), and $f(\cdot) \in L_2(-\pi,\pi)$.

(iii) If $f(\cdot) \in \mathcal{L}_2(-\pi,\pi)$, then $\{c_k\} \in l_2$, the series converges as an element of $L_2(-\pi,\pi)$ (i.e., not necessary for all t).

Theorem 6.19 *(Plancherel's-Parseval's Theorem) Let $f, g \in \mathcal{L}_2(-\pi,\pi)$. Let c_k, d_k be the Fourier coefficients for f and g correspondingly. Then*

$$\sum_{k=-\infty}^{\infty} \overline{c_k} d_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f}(t) g(t) dt.$$

In particular,

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt.$$

Corollary 6.20 *In the notations of the last theorem,*

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

In literature, $\{c_k\}$ is said to be a representation of f in the frequency domain; k is the frequency.

[4] In literature, the interval $[0, 2\pi]$ can be replaced by some other interval, and the multiplier $\frac{1}{\sqrt{2\pi}}$ can be replaced by a different one.

[5] meaning that $\int_{-\pi}^{\pi} |f(t) - \sum_{k=-N}^{N} c_k e^{ikt}|^2 dt \to 0$ as $N \to +\infty$

Fourier series and differentiation

Theorem 6.21 *Let* $\{c_k\} \in l_1$ *and* $\{kc_k\} \in l_1$. *Let* $f(t) = \sum_{k=0}^{\infty} c_k e^{ikt}$ *for some integer* N. *Then* $\frac{df}{dt}(t) = \sum_{k=-\infty}^{\infty} ikc_k e^{ikt}$. *If* $\{k^2 c_k\} \in l_1$, *then* $\frac{d^2 f}{dt^2}(t) = -\sum_{k=-\infty}^{\infty} k^2 c_k e^{ikt}$.

6.4 Z-transform

The Z-transform is based on a modification of Fourier series: it represents dynamic discrete time processes $x_0, x_1, x_2, ...$ as the Fourier coefficients of some function $Y(r) : [0, 2\pi] \to \mathbf{C}$, such that

$$Y(r) = \sum_{t=0}^{\infty} e^{-irt} x_t.$$

Let $T = \{z \in \mathbf{C} : |z| = 1\}$. Let $z \triangleq e^{ir} \in T$, then $e^{-irk} = z^{-k}$. Let $X(z) \triangleq Y(r)$ for $z = e^{ir}$. We have that $X : T \to \mathbf{C}$ is such that

$$X(z) = \sum_{t=0}^{\infty} z^{-t} x_t.$$

This transform is convenient for dynamic discrete time systems. In the terms of signal processing theory, the Z-transform converts a discrete time-domain signal, which is a sequence of real numbers, into a complex frequency-domain representation.

Let Z-transform of $\{x_t\}$ be $X(z)$, let $x_0 = 0$, and let $S_1(x_0, x_1, ...) = (x_1, x_2, x_3, ...)$, then z-transform of $S_1\{x_t\}$ is $zX(z)$.

It can be applied for discrete-time linear equations. For instance, let us consider the following discrete time dynamic system, i.e., the equation for a scalar dynamic discrete time process:

$$\begin{aligned} &x_{t+1} = ax_t + f_t, \quad t = 0, 1, 2, ..., \\ &x_0 = 0. \end{aligned}$$

Let $(y_0, y_1, ...) = (x_1, x_2, ...)$. Then $y_t = ax_t + f_t$,

$$Y(z) = \sum_{t=0}^{+\infty} z^{-t} y_t = \sum_{t=0}^{+\infty} z^{-t} x_{t+1} = z \sum_{t=0}^{+\infty} z^{-(t+1)} x_{t+1} = z \sum_{s=1}^{+\infty} z^{-s} x_s = zX(z),$$

where $X(z) = \sum_{t=0}^{+\infty} z^{-t} x_t$. Therefore, $zX(z) = aX(z) + F(z)$, where $X(z)$, $F(z)$ are Z-transforms of $\{x_t\}$ and $\{f_t\}$ respectively, i.e.,

$$X(z) = \frac{F(z)}{z - a}.$$

Applications in control and system theory

Similarly to the case of continous time processes, in control and system theory, the process $\{f_t\}$ (or $F(z)$) is interpreted as an input of a *linear discrete time system* with *transfer function* $\chi(z) = \frac{1}{z-a}$, i.e., $X(z) = \chi(z)F(z)$. The process $\{x_t\}$ (or $X(z)$) is interpreted as an output of this system. The same model is used for more general dynamic systems. The transfer function describes completely the properties of input-output system. It is why the methods of complex analysis are very common in system theory. For instance the system from our example is "stable"[6] iff $|a| < 1$. In a case of more general χ, the system is stable iff the transfer function does not have singularities outside the unit circle T.

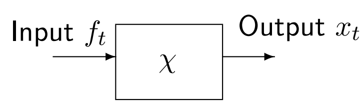

Figure 6.2: The block diagram for the system $x_{t+1} = ax_t + f_t$.

[6]Repeat that one of many possible definitions is that a system is stable if any bounded input produces a bounded output on infinite horizon.