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Optimal investment strategies with bounded risks, general utilities, and goal achieving

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Abstract

This paper investigates an investment/hedging problem in a multi-stock financial market with random appreciation rates. Only those strategies with bounded risks (i.e. they guarantee that a given claim will be replicated with an error not exceeding a given level) are considered. Moreover, admissible strategies are based upon observations of market prices rather than those of the appreciation rates. An optimal strategy, which does not depend on the current estimation of the appreciation rates of the stocks, is obtained for a model with a general utility function. The result is further shown to cover some important special cases, especially the so-called goal achieving problem.

JEL classification: D52; D81; D84; G11

Keywords: Diffusion market model; Optimal portfolio; Bounded risks; Goal achieving

1. Introduction

This paper investigates an investment/risk-hedging problem for a stochastic diffusion model of a security market consisting of a risk-free bond and a finite number of risky stocks. Associated with a given contingent claim, an investment strategy is said to have bounded risk if the claim is replicated with an error not exceeding a given level. In a broad sense, a bounded risk investment strategy is also called a hedging strategy. The most well-known hedging strategies for this model were obtained by Black and Scholes (1973) and Merton (1969, 1973). For the Black and Scholes model, the strategies were used to hedge given claims exactly (without any error). For the Merton model, the strategies were

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obtained for an optimization problem of maximizing $\boldsymbol{E}U(X(T))$, where $X(T)$ is the wealth at the expiration time T and $U(\cdot)$ is a utility function. For various variants and extensions, see, e.g. Samuelson (1969), Hakansson (1971), Perold (1984), Karatzas et al. (1987), Dumas and Liucinao (1991), Zhou (1998), and Khanna and Kulldorff (1999). However, in the literature explicit formulae for optimal strategies have been established only for the cases when appreciation rates of the stocks are non-random and known, and $U(\cdot)$ has quadratic form, log form or power form. In the general case of random appreciation rates, solution of the optimal investment problem calls for using the so-called backward stochastic differential equations (for a most updated account of this theory see Chapter 7 of Yong and Zhou (1999)), which unfortunately is difficult to solve explicitly and computationally.

Another problem of wide interest is a mean–variance hedging, or a problem of minimizing $\boldsymbol{E}|X(T) - \xi|^2$, where ξ is a given random claim. For this problem, explicit solutions were obtained for the case of observable appreciation rates, see, e.g. Föllmer and Sondermann (1986), Duffie and Richardson (1991), Pham et al. (1998), Kohlmann and Zhou (1998), Pham et al. (1998), and Laurent and Pham (1999). The resulting optimal hedging strategies are combinations of the Merton strategy and the Black and Scholes strategy, which depend on the direct observation of the appreciation rates. Unfortunately, as well known the appreciation rates are usually hard to observe in real-time market, especially when the volatility coefficients are larger than the average deviations of the appreciation rates per unit time. Moreover, in these studies the error between the terminal wealth and the claim is bounded in the mean–variance sense, rather than in the almost-surely one.

In this paper, we consider an investment/hedging model with several new features. First of all, in our model we do not assume that the appreciation rates of the stocks are non-random and observable; we only assume that the distributions of the appreciation rates are known based on the observation of the stock prices. The finally derived strategy depends only on the current stock prices, the distributions of the appreciation rates, and one scalar parameter which can be calculated numerically. Second, our model involves the replication of a given claim with a guaranteed error bound (gap). More precisely, our admissible strategies ensure that the replication errors do not exceed a given level almost surely. Note that in the classical problem (for a complete market) of an exact replication, the strategy is uniquely determined by the claim. In an incomplete market, where an exact replication is no longer generally possible, it is sensible to consider replications with some gap, which in turn makes it possible to choose among many possible strategies. Finally, the utility function under consideration in our model is a fairly general one, covering the mean–variance criterion, non-continuous functions, and nonlinear concave functions as special cases. In particular, our general utility function incorporates the so-called goal achieving problem. A goal achieving problem is to maximize the probability that the event of reaching a prescribed goal happens before the event of a failure. Specifically, it is to maximize $\boldsymbol{P}(\tau_1 \geq \tau_2)$, where τ_i is the first time that the discounted process $\hat{X}(t) \triangleq e^{r(T-t)}X(t)$ reaches the level k_i, $i = 1, 2$. Here r is the risk free interest rate, k_2 is the goal level while k_1 is a level considered to be a failure (certainly, $k_1 < k_2$).

The goal achieving problem is interesting in its own right. The problem for a single-stock market model with only additive stochastic disturbances was first solved by Karatzas (1997), where the constructed optimal strategy depends only on the distribution of the stock appreciation rate. In the present paper, using an approach completely different than that of Karatzas

(1997), we are able to derive optimal strategies that are independent of the appreciation rate estimations, for a general model with multiple, correlated stocks and additional constraints of bounded risks. In particular, it is concluded that for an optimal strategy the levels k_1 and k_2 will never be achieved before the expiration time.

It should also be noted that for a general problem with non-random appreciation rates, Khanna and Kulldorff (1999) showed that the so-called Mutual Fund Theorem holds, namely, the optimum can be achieved on a set of Merton type strategies. But, this theorem does not hold for the case of random and non-observable rates that is being studied here, nor does the method of Khanna and Kulldorff (1999) apply.

The rest of this paper is organized as follows. In Section 2, the general model under consideration is formulated and necessary preliminaries are given. In Section 3, an optimal solution to the general problem is presented. Section 4 discusses several important special cases of the general problem, including the goal achieving problem. Numerical results are reported in Section 5. In Section 6, some concluding remarks are given. Finally, in Appendix, proofs of the results are supplied.

2. Problem formulation and preliminaries

Consider a diffusion model of a security market consisting of a risk-free bond with the price $B(t)$, $t \geq 0$, and n risky stocks with prices $S_i(t)$, $t \geq 0$, $i = 1, 2, \ldots, n$, where $n < +\infty$ is given. Throughout this paper all random processes are defined on a standard probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. The price of the bond is given by the following

$$B(t) = \mathrm{e}^{rt} B_0, \tag{2.1}$$

where $r \geq 0$ and B_0 are given constants. On the other hand, the prices of the stocks evolve according to the following stochastic differential equations

$$\begin{cases} \mathrm{d}S_i(t) = S_i(t)\left(a_i\,\mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij}\,\mathrm{d}w_j(t)\right), \quad t > 0, \\ S_i(0) = S_{i0}, \end{cases} \tag{2.2}$$

where $w(t) \equiv (w_1(t), \ldots, w_n(t))$ is a standard n-dimensional Wiener process (with $w(0) = 0$), a_i is the (random) appreciation rate of the ith stock, and σ_{ij} the volatility coefficient. The initial price $S_{i0} > 0$ is a given non-random constant. We set $a = (a_1, \ldots, a_n)$ and $S(t) = (S_1(t), \ldots, S_n(t))$. In addition, we introduce an auxiliary process $S_*(t)$ which is defined as the price process $S(t)$ under the assumption that the appreciation rates of all the stocks coincide with the bond rate, i.e. $a_i(\omega) \equiv r$, $\forall i$. Note that, we choose to use this process rather than a risk-neutral probability measure because it is more convenient for our aims, as evident from the sequel.

Remark that in practice, the volatility coefficients σ_{ij} can be effectively estimated from $S_i(t)$. In contrast, it is more difficult to estimate the appreciation rate a_i. In fact, an estimator of a_i is not satisfactory when the volatility is sufficiently large. In view of these, we assume that $\sigma \triangleq (\sigma_{ij})$ is a known, deterministic $n \times n$ matrix such that $\sigma\sigma' > 0$; yet a is random and not directly observable. Moreover, we assume that a is independent of $w(\cdot)$, and $\boldsymbol{P}(|a| \leq c) = 1$ for a constant $c > 0$.

Let $\mathcal{F}_t^S \subset \mathcal{F}$ be the right-continuous monotonically increasing filtration of complete σ-algebras generated by $S(t)$, $t \geq 0$.

Let $X(t)$ be the wealth of an agent at time t with $X(0) > 0$ being the initial wealth. Then

$$X(t) = \beta(t)B(t) + \sum_{i=1}^{n}\gamma_i(t)S_i(t), \quad t \geq 0, \tag{2.3}$$

where $\beta(t)$ is the quantity of the bond, and $\gamma_i(t)$ the quantity of the ith stock in the portfolio. Let $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$. It is clear that the pairs $(\beta(t), \gamma(t))$ describe the state of the portfolio at time t. We call them strategies.

Definition 2.1. A pair $(\beta(\cdot), \gamma(\cdot))$ is said to be an admissible strategy if $\beta(t)$, $\gamma_i(t)$, and $\gamma_i(t)S_i(t)$, $i = 1, \ldots, n$, are random processes that are progressively measurable with respect to the filtration $\mathcal{F}_t^S$ and satisfy

$$\boldsymbol{E}\int_0^T |\beta(t)|^2\,\mathrm{d}t < +\infty, \quad \sum_{i=1}^{n}\boldsymbol{E}\int_0^T |\gamma_i(t)S_i(t)|^2\,\mathrm{d}t < +\infty, \quad \forall T > 0.$$

Definition 2.2. An admissible strategy $(\beta(t), \gamma(t))$ is said to be self-financing, if

$$\mathrm{d}X(t) = \beta(t)\,\mathrm{d}B(t) + \sum_{i=1}^{n}\gamma_i(t)\,\mathrm{d}S_i(t). \tag{2.4}$$

By definition, any self-financing admissible strategy has the form

$$\gamma(t) = \Gamma(t, S(\cdot)|_{[0,t]}), \qquad \beta(t) = \frac{X(t) - \sum_{i=1}^{n}\gamma_i(t)S_i(t)}{B(t)}, \tag{2.5}$$

where $\Gamma(t,\cdot) : C([0,t]; \boldsymbol{R}^n) \to \boldsymbol{R}^n$ is a functional, $t \geq 0$, and $X(t)$ is defined through the closed system (2.4) and (2.5). Notice that the random processes $(\beta(t), \gamma(t))$ with a same $\Gamma(\cdot,\cdot)$ in (2.5) may be different with different $a(\cdot)$.

Let the initial wealth $X(0)$ be fixed. For a self-financing admissible strategy $(\beta(\cdot), \gamma(\cdot))$, $\beta(t)$ and $X(t)$ are uniquely determined by $\gamma(\cdot)$. We shall then denote by $X(t, \gamma(\cdot))$ the corresponding total wealth at time t.

Definition 2.3. Let the initial wealth $X(0)$ and a time $T > 0$ be fixed, and $\xi = \phi(S(\cdot)|_{t\in[0,T]})$, where $\phi : C([0,T]; \boldsymbol{R}^n) \to \boldsymbol{R}$ is a measurable functional. A self-financing admissible strategy $(\beta(\cdot), \gamma(\cdot))$ is said to replicate the claim ξ with the initial wealth $X(0)$ if

$$X(T, \gamma(\cdot)) = \xi, \quad \text{a.s.}$$

Definition 2.4. Let $X(0)$ and $T > 0$ be fixed, and ξ_1, ξ_2 be random numbers such that $-\infty \leq \xi_1 \leq \xi_2 \leq +\infty$, a.s.. An admissible strategy $(\beta(\cdot), \gamma(\cdot))$ is said to be a bounded risk strategy with bounds ξ_1, ξ_2 if

$$\xi_1 \leq X(T, \gamma(\cdot)) \leq \xi_2, \quad \text{a.s.}$$

Let $\boldsymbol{R}_+^n \triangleq \{x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{R}^n : x_i \geq 0, i = 1, 2, \ldots, n\}$, and $\mathring{\boldsymbol{R}}_+^n \triangleq \{x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{R}^n : x_i > 0, i = 1, 2, \ldots, n\}$.

Definition 2.5. A function $C(\cdot)\colon \mathring{\boldsymbol{R}}^n_+ \to \boldsymbol{R}$ is said to be of polynomial growth if there exist constants $c_1 > 0$ and $c > 0$ such that

$$|C(x)| \le c_1 \left(|x|^c + \sum_{i=1}^{n} x_i^{-c} + 1 \right), \quad \forall x = (x_1, \dots, x_n) \in \mathring{\boldsymbol{R}}^n_+.$$

Now we formulate the problem to be studied in this paper. Let $T > 0$ and $X(0)$ be fixed, ξ_1 and ξ_2 be given random numbers, and $U(\cdot)\colon \boldsymbol{R} \to \boldsymbol{R}$ be a given utility function such that there exist constants $c_1 > 0$ and $c > 0$ satisfying

$$U(x) \le c_1(|x|^c + 1), \quad \forall x \in \boldsymbol{R}.$$

The problem is to find a self-financing admissible strategy $(\beta(\cdot), \gamma(\cdot))$ that solves the following optimization problem:

$$\begin{array}{ll} \text{Maximize} & \boldsymbol{E} U(X(T, \gamma(\cdot))), \\ \text{Subject to} & \begin{cases} X(0, \gamma(\cdot)) = X(0), \\ \xi_1 \le X(T, \gamma(\cdot)) \le \xi_2, \quad \text{a.s.} \end{cases} \end{array} \tag{2.6}$$

Throughout this paper we impose the following assumptions.

Assumption 2.1. The given random bounds ξ_1 and ξ_2 satisfy the following:

1. $\xi_i = h_i(S(T))$, $i = 1, 2$, where the functions $h_i\colon \mathring{\boldsymbol{R}}^n_+ \to [-\infty, +\infty]$ are such that whenever $h\colon \mathring{\boldsymbol{R}}^n_+ \to \boldsymbol{R}$ is a function of polynomial growth, so are $\max(h_1(x), h(x))$ and $\min(h_2(x), h(x))$.
2. $-\infty \le h_1(x) \le h_2(x) \le +\infty$ for all x.
3. $\boldsymbol{E} h_1(S_*(T)) < \mathrm{e}^{rT} X(0) < \boldsymbol{E} h_2(S_*(T))$.

Assumption 2.2. There exist constants $q_0 \in \{-\infty, 0\}$, $c_1 > 0$, $c > 0$, measurable sets $I \subseteq (q_0, +\infty)$ and $I_0 \subseteq I$, and a function $F(\cdot,\cdot) : \boldsymbol{R} \times \mathring{\boldsymbol{R}}^n_+ \to \boldsymbol{R}$ such that the following hold:

1. $\mathrm{mes}((q_0, +\infty) \backslash I) = 0$.
2. For any $x \in \boldsymbol{R}^n$ and $q \in I$, the optimization problem

$$\begin{array}{ll} \text{Maximize} & U(y) - qy, \\ \text{Subject to} & y \in [h_1(x), h_2(x)] \end{array} \tag{2.7}$$

has a solution $y = F(q, x) > q_0$, and

$$|F(q,x)| \le c_1 \left(|q|^c + |q|^{-c} + |x|^c + \sum_{i=1}^{n} x_i^{-c} + 1 \right), \quad \forall x = (x_1, \dots, x_n) \in \mathring{\boldsymbol{R}}^n_+.$$

Moreover, if $q \in I_0$ then the solution of (2.7) is unique.

It is shown below that Assumptions 2.1 and 2.2 are satisfied in many important special cases.

3. Main results

First we need to introduce some notation. Define the set $\mathcal{A} = \{\alpha \in \boldsymbol{R}^n : |\alpha| \leq c\}$, where c is a constant such that $\boldsymbol{P}(|a| \leq c) = 1$. Without loss of generality, we may take the probability space as follows: $\Omega = \mathcal{A} \times \Omega_c$, where $\Omega_c = C([0, T]; \boldsymbol{R}^n)$. Let $\mathcal{B}$ be the σ-algebra of subsets of Ω_c generated by the cylinder sets. Furthermore, we assume that there are σ-additive probability measures $\nu(\cdot)$ and $\boldsymbol{P}_c$ on $\mathcal{A}$ and Ω_c, respectively, such that $P = \nu \times \boldsymbol{P}_c$.

Let $\vec{e} \triangleq (1, \dots, 1)' \in \boldsymbol{R}^n$, $\vec{r} \triangleq r\vec{e} \in \boldsymbol{R}^n$, $\tilde{a} \triangleq a - \vec{r}$.. Let $\boldsymbol{S}(t) \triangleq \operatorname{diag}(S_1(t), \dots, S_n(t))$ be the diagonal matrix with the diagonal elements defined by (2.2). Furthermore, let $\boldsymbol{S}_*(t) \triangleq \operatorname{diag}(S_{*1}(t), \dots, S_{*n}(t))$ be the diagonal matrix with the diagonal elements defined by (2.2) with $a = \vec{r}$.

Finally, introduce the Banach space $\mathcal{Y}^1$ of functions $u(\cdot, \cdot) : \mathring{\boldsymbol{R}}^n_+ \times [0, T] \to \boldsymbol{R}$ satisfying

$$\sup_t \boldsymbol{E}|u(S(t), t)|^2 + \sum_{k=1}^n \boldsymbol{E} \int_0^T \left| \frac{\partial u}{\partial x_k}(S(t), t) S_k(t) \right|^2 \, \mathrm{d}t < +\infty,$$

with the norm

$$||u(\cdot, \cdot)||_{\dagger^1} \triangleq \left(\sup_t \boldsymbol{E}|u(S(t), t)|^2 + \sum_{k=1}^n \boldsymbol{E} \int_0^T \left| \frac{\partial u}{\partial x_k}(S(t), t) S_k(t) \right|^2 \, \mathrm{d}t \right)^{1/2}.$$

Actually, the above space is a weighted Sobolev space. Hence the derivatives involved are in the sense of distributions.

The following lemma is an adaptation of the standard result in the Black and Scholes theory of replicating claims (see Black and Scholes (1973)) to the class of $\mathcal{F}_t^S$-adapted strategies introduced above. For the case when $a = a(t)$ is adapted to the filtration generated by $w(t)$, a corresponding result is presented in Karatzas (1996), Theorem 1.2.1. However, in our case the assumption of that theorem is not satisfied, since a is independent of $w(\cdot)$.

Lemma 3.1. *Let $f(\cdot) : \mathring{\boldsymbol{R}}^n_+ \to \boldsymbol{R}$ be a measurable function such that $\boldsymbol{E} f(S_*(T))^2 < +\infty$ and $\boldsymbol{E} f(S(T))^2 < +\infty$. Then a self-financing admissible strategy $(\beta(\cdot), \gamma(\cdot))$ which replicates the claim f(S(T)) exists if and only if $\boldsymbol{E} f(S_*(T)) = e^{rT} X(0)$. Moreover, when the replication $(\beta(\cdot), \gamma(\cdot))$ exists, $\gamma(t)$ and the corresponding X(t) are given by*

$$\gamma_i(t) = \frac{\partial H}{\partial x_i}(\mathrm{e}^{r(T-t)} S(t), t), \qquad X(t) = \mathrm{e}^{r(t-T)} H(\mathrm{e}^{r(T-t)} S(t), t),$$

where the function $H(\cdot, \cdot) : \mathring{\boldsymbol{R}}^n_+ \times [0, T] \to \boldsymbol{R}$ is the solution to the following Cauchy problem:

$$\begin{cases} \dfrac{\partial H}{\partial t}(x, t) + \dfrac{1}{2} \displaystyle\sum_{i,j=1}^n \left[\sum_{k=1}^n (\sigma_{ik}\sigma_{jk}) x_i x_j \frac{\partial^2 H}{\partial x_i \partial x_j}(x, t) \right] = 0, \\ H(x, T) = f(x). \end{cases} \tag{3.1}$$

In addition, the problem (3.1) *admits a solution in the class* $\mathcal{Y}^1$, *and the corresponding processes* $S_i(t)\gamma_i(t)$, $\beta(t)$, *and* $X(t)$ *are all square integrable*.

Note that Eq. (3.1) is in the sense of Sobolev generalized functions.

It is easy to see that

$$\ln S_i(t) = a_i t - \frac{t}{2}\sum_{j=1}^{n}\sigma_{ij}^2 + \sum_{j=1}^{n}\sigma_{ij}w_j(t), \quad \ln S_{i*}(t) = rt - \frac{t}{2}\sum_{j=1}^{n}\sigma_{ij}^2 + \sum_{j=1}^{n}\sigma_{ij}w_j(t).$$

From these formulas, it follows that $S(t)$ has a conditional log-normal probability density function given a, while $S_*(t)$ has an unconditional log-normal probability density function. In fact, $S(t)$ also has a probability density function.

Let $p(x,t)$ and $p_*(x,t)$ be the probability density functions of $S(t)$ and $S_*(t)$, respectively. Define the functions $\psi(x): \mathring{\boldsymbol{R}}^n_+ \to \boldsymbol{R}$ and $f(\lambda, x)$: $(q_0, +\infty) \times \mathring{\boldsymbol{R}}^n_+ \to \boldsymbol{R}$ as follows

$$\psi(x) \triangleq \frac{p(x,T)}{p_*(x,T)}, \qquad f(\lambda, x) \triangleq F\left(\frac{\lambda}{\psi(x)}, x\right). \tag{3.2}$$

Clearly $\psi(x) > 0, \ \forall x$.

Let $H(x,t,\lambda)$: $\mathring{\boldsymbol{R}}^n_+ \times \mathring{\boldsymbol{R}} \times \mathring{\boldsymbol{R}}_+ \to \boldsymbol{R}$ be the solution of the Eq. (3.1) with the following terminal condition (parameterized by λ):

$$H(x,T,\lambda) = f(\lambda, x).$$

Theorem 3.1. *We have* $\boldsymbol{E}|f(\lambda, S_*(T))|^2 < +\infty$ *and* $\boldsymbol{E}|f(\lambda, S(T))|^2 < +\infty$ *for all* $\lambda \neq 0$, *and the Eq.* (3.1) *admits a solution for the terminal condition* $f = f(\lambda, x)$ *with any* $\lambda \neq 0$. *Furthermore, assume that*

$$\text{mes}\left\{x \in \mathring{\boldsymbol{R}}^n_+ : \ \frac{\lambda}{\psi(x)} \notin I\right\} = 0, \quad \forall \lambda > q_0, \quad \lambda \neq 0, \tag{3.3}$$

and there exists $\hat{\lambda} > q_0$, $\hat{\lambda} \neq 0$, *such that*

$$\boldsymbol{E} f(\hat{\lambda}, S_*(T)) = e^{rT} X(0). \tag{3.4}$$

Then the self-financing admissible strategy $(\gamma(\cdot), \beta(\cdot))$ *that replicates the claim* $f(S(T), \hat{\lambda})$ *is an optimal solution of the problem (2.6). The corresponding strategy is*

$$\gamma_i(t) = \frac{\partial H}{\partial x_i}(e^{r(T-t)}S(t), t, \hat{\lambda}), \qquad \beta(t) = \frac{X(t) - \sum_{i=1}^{n}\gamma_i(t)S_i(t)}{B(t)}, \tag{3.5}$$

where X(t) is the corresponding wealth given by

$$X(t) = e^{r(t-T)} H(e^{r(T-t)}S(t), t, \hat{\lambda}). \tag{3.6}$$

Moreover, any optimal solution must be represented via (3.5) *and* (3.6), *where* $\hat{\lambda} > q_0$, $\lambda \neq 0$, *is such that* (3.4) *holds. Furthermore, if*

$$\text{mes}\left\{x \in \mathring{\boldsymbol{R}}^n_+ : \quad \frac{\lambda}{\psi(x)} \notin I_0\right\} = 0, \quad \forall \lambda > q_0, \quad \lambda \neq 0, \tag{3.7}$$

then the optimal solution is unique in the sense that all optimal processes X(t), γ(t) and β(t) are the same (equivalent) even for possibly different $\hat{\lambda}$ satisfying (3.4).

Notice that

$$H(x,t,\lambda)=\int_{\boldsymbol{R}_+^n}\bar{p}_*(y,T,x,t)f(\lambda,y)\,\mathrm{d}y=\boldsymbol{E}\{f(\lambda,S_*(T))/S_*(T)=x\}, \tag{3.8}$$

where $\bar{p}_*(x,t,y,\tau)$ is the conditional probability density function for the vector $S_*(t)$ given the condition $S_*(\tau)=y, 0\le\tau<t\le T$. In particular, $p_*(x,t)=\bar{p}_*(x,t,S(0),0)$, and (3.4) has the form

$$\int_{\boldsymbol{R}_+^n}p_*(x,T)f(\hat{\lambda},x)\,\mathrm{d}x=\mathrm{e}^{rT}X(0).$$

Remark 3.1. We can see from Theorem 3.1 that the optimal strategy depends only on the distribution of the appreciation rates, not these rates themselves.

Remark 3.2. Karatzas (1997) studied a simpler model where $n=1$, $\mathrm{d}S(t)=a\,\mathrm{d}t+\sigma\,\mathrm{d}w(t)$, with σ being a constant, and a being a random number independent of $w(\cdot)$. A problem of maximizing the probability of achieving a certain level was solved. It was shown that for this simplest model, the optimal strategy does not depend on the current estimation of the appreciation rate a. In the next section we will demonstrate that the goal achieving problem is a special case of the problem (2.6), and the independence on the estimation of the appreciate rates is indeed a property possessed by an optimal strategy even for more general model.

Next, we give a sufficient condition for the critical Eq. (3.4) to have a solution $\hat{\lambda}>q_0$.

Lemma 3.2. *Let* $h_i(\cdot):\ \mathring{\boldsymbol{R}}_+^n\to\boldsymbol{R},\ i=1,2$, *be functions of polynomial order of growth, and let* q_0 *and F(q, x) be as specified in Assumption* 2.2. *If either*

$$F(q,x)\to h_1(x)\quad \text{as}\, q\to q_0,\quad \text{and}\quad F(q,x)\to h_2(x)\quad \text{as}\, q\to+\infty,\quad \forall x$$

or

$$F(q,x)\to h_2(x)\quad \text{as}\, q\to q_0,\quad \text{and}\quad F(q,x)\to h_1(x)\quad \text{as}\, q\to+\infty,\quad \forall x,$$

then there exists $\hat{\lambda}>q_0$ *such that* (3.4) *holds*.

Notice that for the most important special cases to be discussed below, $\int_{\boldsymbol{R}_+^n}p_*(x,T)f(\lambda,x)\,\mathrm{d}x$ is a monotonically decreasing function of λ. Hence as long as $\psi(x)$ is known, $\hat{\lambda}$ can be numerically determined from (3.4). On the other hand, an explicit formula for $\psi(x)$ is derived below in the case of non-correlated stocks (i.e. $\sigma_{ij}=0$, $\forall i\ne j$).

To this end, for $i=1,\dots,n$, let

$$\bar{\sigma}_i\triangleq\sqrt{T}\sigma_{ii},\qquad \mu_i=(a_i-r)T,\qquad \theta_i\triangleq\frac{\mu_i(\bar{\sigma}_i^2-2rT)-\mu_i^2}{2\bar{\sigma}_i^2}. \tag{3.9}$$

It is easy to see that if $\sigma_{ij} = 0 \quad \forall i \neq j$, then

$$\bar{p}_*(y, \tau, x, t) = \prod_{i=1}^{n} p_*^{(i)}(y_i, \tau, x_i, t), \tag{3.10}$$

where $x = (x_1 \ldots, x_n)$, $y = (y_1, \ldots., y_n)$, and

$$p_*^{(i)}(y_i, \tau, x_i, t) = \frac{1}{x_i \sigma_{ii}\sqrt{2\pi(t-\tau)}} \times \exp \frac{-(\ln(y_i) - \ln(x_i) - r(\tau - t) + \sigma_{ii}^2(t-\tau)/2)^2}{2\sigma_{ii}^2(t-\tau)}.$$

Theorem 3.2. *If $\sigma_{ij} = 0 \quad \forall i \neq j$ then*

$$\psi(x) = \frac{p(x, T)}{p_*(x, T)} = \boldsymbol{E}\left[\prod_{i=1}^{n}\left(\frac{x_i}{S_i(0)}\right)^{\mu_i/\bar{\sigma}_i^2} \mathrm{e}^{\theta_i}\right], \quad x = (x_1, \ldots, x_n). \tag{3.11}$$

4. Special cases

In this section, we study some important special cases of the general model presented in the previous section.

4.1. Goal achieving problem

Let k_1, k_2 be such that $-\infty < k_1 < k_2 < +\infty$. For any admissible self-financing strategy $(\beta(\cdot), \gamma(\cdot))$, introduce the goal achieving stopping times

$$\tau_1 \triangleq T \wedge \inf\{t : e^{r(T-t)} X(t, \gamma(\cdot)) = k_1\}, \qquad \tau_2 \triangleq \inf\{t : \mathrm{e}^{r(T-t)} X(t, \gamma(\cdot)) = k_2\}.$$

Let $X(0)$ be an initial wealth with $k_1 < \mathrm{e}^{rT} X(0) < k_2$.
Consider the following goal achieving problem:

$$\text{Maximize} \quad \boldsymbol{P}(\tau_1 \geq \tau_2) \quad \text{over } (\beta(\cdot), \gamma(\cdot)) \tag{4.1}$$

$$\text{Subject to} \quad X(0, \gamma(\cdot)) = X(0). \tag{4.2}$$

We now show that this problem is a particular case of the problem (2.6).

Proposition 4.1. *Let $h_1(x) \equiv k_1, h_2(x) \equiv k_2, U(x) = \chi_{\{k_2 \leq x\}}$ where χ is the indicator function. Then Assumptions* 2.1 *and* 2.2 *hold with*

$$q_0 = 0, \quad I = (0, +\infty), \quad I_0 = \{q > 0 : 1 - qk_2 \neq -qk_1\},$$

$$F(q, x) = \begin{cases} k_2, & \text{if } 1 - qk_2 \geq -qk_1, \\ k_1, & \text{if } 1 - qk_2 < -qk_1. \end{cases} \tag{4.3}$$

Furthermore, the assumptions of Theorem 3.1 *hold.*

Thus, by definition,

$$f(\lambda, x) = \begin{cases} k_2, & \text{if } \psi(x) \geq \lambda(k_2 - k_1) \\ k_1, & \text{if } \psi(x) < \lambda(k_2 - k_1) \end{cases} \tag{4.4}$$

and

$$\begin{aligned} H(x, t, \lambda) &= k_1 + (k_2 - k_1)\boldsymbol{P}(\psi(S_*(T)) \geq \lambda(k_2 - k_1)|S_*(t) = x) \\ &= k_1 + (k_2 - k_1)\int_{\boldsymbol{R}^n_+} \bar{p}_*(y, T, x, t)\chi_{\{\psi(y) \geq \lambda(k_2 - k_1)\}}\,\mathrm{d}y, \end{aligned} \tag{4.5}$$

where $\bar{p}_*(y, T, x, t)$ is the probability density function for $S_*(T)$ conditional on $S_*(t) = x$. For the case when $\sigma_{ij} = 0$ for $i \neq j$, the functions $\psi(\cdot)$ and $\bar{p}_*(y, T, x, t)$ are defined explicitly in (3.10) and (3.11).

The proof of Proposition 4.1 is straightforward, hence omitted here.

Theorem 4.1. *The problem* (2.6) *with parameters specified in Proposition* 4.1 *and the Problem* (4.1) *and* (4.2) *have the same optimal value of the functionals to be maximized. More-over, an optimal strategy (as given in Theorem 3.1) for the Problem* (2.6) *with parameters specified in Proposition 4.1 is also optimal for the problem* (4.1) *and* (4.2).

The above proof also leads to the following result immediately.

Corollary 4.1. *Under the assumptions of Proposition* 4.1, *any optimal strategy for the Problem* (4.1) *and* (4.2) *must satisfy* $\boldsymbol{P}(\tau_1 \wedge \tau_2 < T) = 0$.

Corollary 4.1 shows that for an optimal strategy the first time when the wealth achieves k_1 or k_2 occurs only at $t = T$. In other words, stopping the investment before the expiration time T cannot be optimal.

4.2. Mean–variance criteria

The following proposition is devoted to the problem which is close to the Markowitz formulation of mean-variance optimal portfolio selection (see Markowitz (1952)), where the expectation of a return is to be maximized and the dispersion of the return is to be minimized.

Proposition 4.2. *Let* $\xi_1 \equiv -\infty$, $\xi_2 \equiv +\infty$, $U(x) = -kx^2 + cx$, where $c \in \boldsymbol{R}$, $k \in \boldsymbol{R}$, $k > 0$, $c \geq 0$. *Then Assumptions* 2.1 *and* 2.2 *hold with*

$$q_0 = -\infty, \quad I = I_0 = (-\infty, +\infty), \quad F(q, x) = \frac{c - q}{2k}.$$

In this case, the Eq. (3.4) *has an unique solution*

$$\begin{aligned} \hat{\lambda} &= 2k\left(\frac{c}{2k} - \mathrm{e}^{rT}X(0)\right)\left(\int_{\boldsymbol{R}^n_+} \frac{p_*(x, T)^2}{p(x, T)}\,\mathrm{d}x\right)^{-1} \\ &= 2k\left(\frac{c}{2k} - \mathrm{e}^{rT}X(0)\right)\left(\boldsymbol{E}\frac{1}{\psi(S_*(T))}\right)^{-1}. \end{aligned} \tag{4.6}$$

Again, the above result can be verified directly. Moreover, it is shown in Lemma A.2 in Appendix that the integrand in (4.6) is integrable.

4.3. Nonlinear concave utility functions

Proposition 4.3. *Let $U(x) : \mathring{\boldsymbol{R}}_+ \to \boldsymbol{R}$ be a concave differentiable function such that $U'(x) : (0, +\infty) \to (0, +\infty)$ is a bijection (i.e. a one-to-one mapping), and there exist constants $C > 0$, $0 < c_1 < 1$, and $c_2 > 0$ satisfying*

$$|U(x)| \leq C(x^{c_1} + x^{-c_2} + 1), \quad |V(x)| \leq C(x^{c_2} + x^{-c_2} + 1), \quad \forall x > 0, \tag{4.7}$$

where V(x) is the converse function of U′(x). Then Assumption 2.2 holds with

$$q_0 = 0, \quad I = I_0 = (0, +\infty), \quad F(q, x) = \begin{cases} h_1(x), & \text{if } V(q) \leq h_1(x), \\ V(q), & \text{if } h_1(x) < V(q) < h_2(x), \\ h_2(x), & \text{if } V(q) \geq h_2(x). \end{cases} \tag{4.8}$$

In this case, the Eq. (3.4) *has a unique solution. Furthermore, if $h_1(x) \leq 0$, $\forall x$, $h_2(x) \equiv +\infty$, and $V(x) = Kx^{-k}$, where $K > 0$, $k > 0$ are constants, then*

$$\hat{\lambda} = (\mathrm{e}^{rT} X(0))^{-1/k} \left(K \int_{\boldsymbol{R}_+^n} p_*(x, T)^{1-k} p(x, T)^k \, \mathrm{d}x \right)^{1/k}. \tag{4.9}$$

The proof is omitted here as it can be checked directly.

Notice if $U(x) = \ln(x)$, then $V(x) = x^{-1}$; if $U(x) = x^{1/\delta}$, $\delta > 1$, then $V(x) = (\delta x)^{-\delta'}$, where $\delta' = \delta(\delta - 1)^{-1}$.

Also, it is a direct consequence of Lemma A.2 in Appendix that the integrand in (4.9) is integrable.

5. Illustrative examples

Example 5.1. We present a numerical solution of the goal achieving problem (4.1) and (4.2), with the following parameters: $n = 1$, $S(0) = 1.6487$, $r = 0$, $X(0) = 1$, $T = 1$, $\sigma = 0.5$, $k_2 = 1.2$, $k_1 = 1/1.2 = 0.8333$, $\boldsymbol{P}(a(t) \equiv \alpha_1) = \boldsymbol{P}(a(t) \equiv \alpha_2) = 1/2$, where $\alpha_1 = 0.2$, $\alpha_2 = \log(2 - \mathrm{e}^{0.2})$ (under this assumption, $\boldsymbol{E}S(T) = S(0)$).

With these parameters, optimal claim $f(\hat{\lambda}, x)$ is given by the formula

$$f(\hat{\lambda}, x) = \begin{cases} \dfrac{1}{1.2} & \text{if } x \in (1.1070, 2.3490) \\ 1.2 & \text{if } x \notin (1.1070, 2.3490) \end{cases}$$

with $\hat{\lambda} = 2.5915$.

Furthermore, by (3.6) and (3.8), we have

$$X(t) = H(S(t), t), \tag{5.1}$$

where

$$H(x,t) = \frac{1}{1.2} + \left(1.2 - \frac{1}{1.2}\right) \times \left(\int_0^{1.1070} \bar{p}_*(y,T,t,x)\,\mathrm{d}y + \int_{2.3490}^{+\infty} \bar{p}_*(y,T,t,x)\,\mathrm{d}y \right),$$

$$\bar{p}_*(y,T,x,t) = \frac{1}{y\sigma\sqrt{2\pi(T-t)}} \exp \frac{-(\ln(y) - \ln(x) - r(T-t) + \sigma^2(T-t)/2)^2}{2(T-t)\sigma^2}. \tag{5.3}$$

The strategy can be easily calculated from (3.5), (5.2) and (5.3).

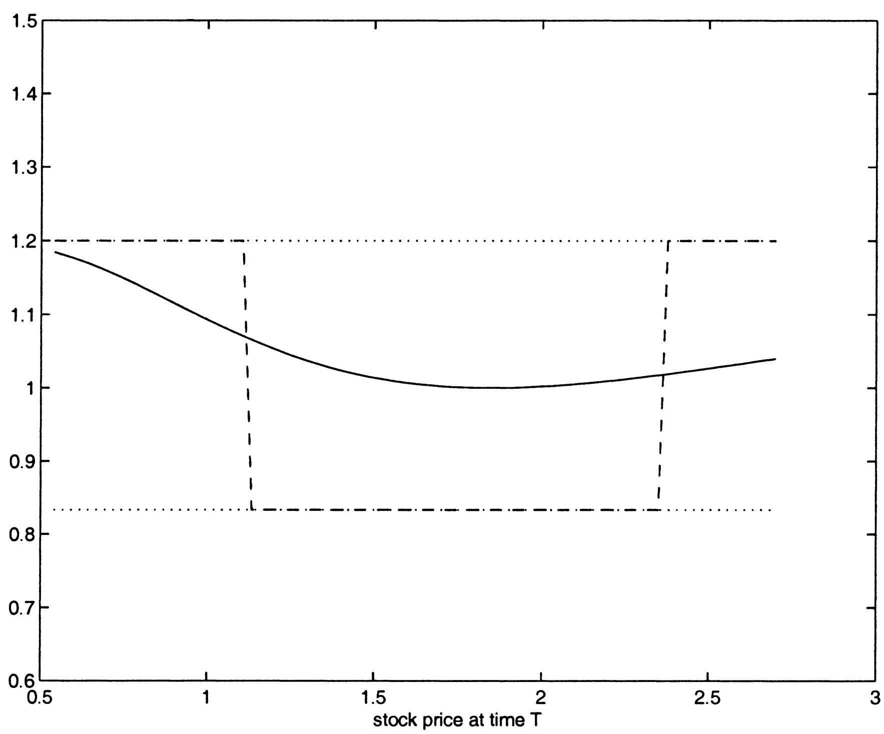

Fig. 1. Optimal claim $f(\hat{\lambda}, x)$ and $H(x, 0)$ for goal achieving with $k_1 = 1/1.2$, $k_2 = 1.2$. (—): values of $H(x, 0)$; (- - -): values of $f(\hat{\lambda}, x) = H(x, T)$.

The wealth process associated with the optimal strategy is given by the following:

$$X(t) = 0.8 + 0.3666[1 - \boldsymbol{P}(S_*(T) \in (1.1070, 2.3490)|S(t))].$$

Fig. 1 shows $H(x,0)$ and the optimal claim $f(\hat{\lambda}, x) = H(x, T)$.

Example 5.2. Consider a problem of optimal replication of a European put option with a possible gap. This problem is a particular case of the problem (2.6). We present a numerical solution with the following parameters: $n = 1$, $U(x) = \ln(x)$, $S(0) = 1.6487$, $r = 0$, $X(0) = 1$, $T = 1$, $\sigma = 0.5$,

$$h_1(x) \equiv (S(0) - x)^+, \qquad h_2(x) = \begin{cases} (2S(0) - x)^+ & \text{if } x \le 1.8S(0) \\ 0.2S(0) & \text{if } x > 1.8S(0), \end{cases}$$

$\boldsymbol{P}(a(t)\equiv\alpha_1) = \boldsymbol{P}(a(t)\equiv\alpha_2) = 1/2$, where $\alpha_1 = 0.2$, $\alpha_2 = \log(2 - e^{0.2})$.

With these parameters, we have $\boldsymbol{E}h_1(S_*(T)) = 0.3255$, $\boldsymbol{E}h_2(S_*(T)) = 1.717$, and the optimal claim $f(\hat{\lambda}, x)$ given by the formula

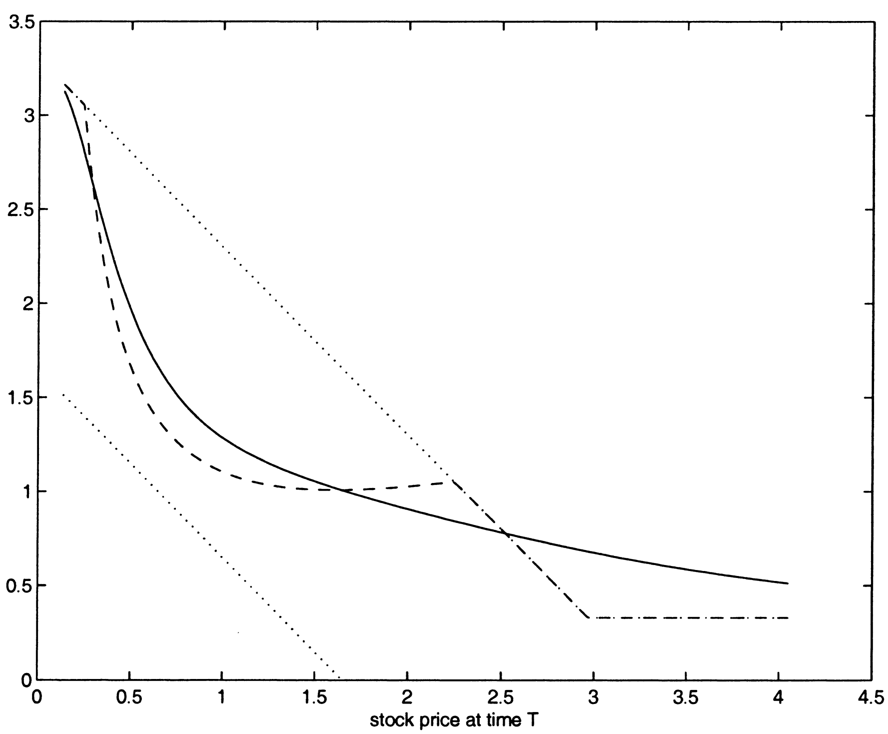

Fig. 2. Optimal claim $f(\hat{\lambda}, x)$ and $H(x, 0)$ for replication of put option with gap. ($\cdots$): values of $h_1(x)$, $h_2(x)$; (——): values of $H(x, 0)$; (- - -): values of $f(\hat{\lambda}, x) = H(x, T)$.

$$f(\hat{\lambda}, x) = \begin{cases} h_1(x), & \text{if } \dfrac{\psi(x)}{\hat{\lambda}} \le h_1(x) \\ \dfrac{\psi(x)}{\hat{\lambda}}, & \text{if } h_1(x) < \dfrac{\psi(x)}{\hat{\lambda}} < h_2(x) \\ h_2(x), & \text{if } \dfrac{\psi(x)}{\hat{\lambda}} \ge h_2(x) \end{cases}$$

with $\hat{\lambda} = 0.8923$. The function $\psi(x)$ is defined in Theorem 3.2:

$$\psi(x) = \frac{1}{2}\sum_{i=1}^{2}\left(\frac{x}{S(0)}\right)^{4\alpha_i} \exp\left(\frac{\alpha_i}{2} - 2\alpha_i^2\right).$$

Therefore, the strategy can be calculated by virtue of (3.5).

Fig. 2 depicts $H(x,0)$ and the optimal claim $f(\hat{\lambda}, x) = H(x, T)$.

6. Conclusions

In this paper we established an optimal investment/hedging model for a multi-stock market that incorporates an additional constraint of bounded risks and a general utility function, and derived a general optimal strategy. It is assumed that the appreciation rates of the stocks are random processes that are independent of the underlying Wiener process. Interestingly, the estimations of appreciation rates were shown to be not a part of the optimal strategy. As a special case, the solution of a goal achieving problem was obtained. This solution was shown to have an interesting feature that the goal can not be achieved before the expiration time.

It seems from the proofs in this paper (see Appendix below) that estimations of volatilities should be included in the optimal strategy for a model with random volatilities. Moreover, for a case when the appreciation rates do depend on the underlying Wiener process in (2.2), estimations of the appreciation rates also should be included in the optimal strategy. Studies on these problems are currently being carried out.

7. Appendix: proofs

In this Appendix, we supply proofs of all the results of this paper.

Proof of Lemma 3.1. Suppose there exists an admissible self-financing strategy $(\beta(\cdot), \gamma(\cdot))$ such that $X(T, \gamma(\cdot)) = f(S(T))$ a.s. We now show that $X(0) = \mathrm{e}^{-rT}\boldsymbol{E} f(S_*(T))$. Denote $X(\cdot) = X(\cdot, \gamma(\cdot))$, and let

$$\tilde{X}(t) \triangleq \mathrm{e}^{r(T-t)}X(t), \quad \tilde{S}(t) \triangleq \mathrm{e}^{r(T-t)}S(t), \quad \tilde{B}(t) \triangleq \mathrm{e}^{r(T-t)}B(t).$$

It follows from the Girsanov theorem (see, e.g. Gihman and Skorohod (1979)) that there exists a probabilistic measure $\boldsymbol{P}_*$ such that $\tilde{S}(t)$ is a martingale. Let $\boldsymbol{E}_*$ be the corresponding expectation. Clearly, $\tilde{X}(0) = \mathrm{e}^{rT}X(0)$, $\tilde{X}(T) = f(S(T))$, and

$$\begin{aligned}\mathrm{d}\tilde{X}(t) &= -r\tilde{X}(t)\,\mathrm{d}t + \mathrm{e}^{r(T-t)}(\gamma(t)\,\mathrm{d}S(t) + \beta(t)\,\mathrm{d}B(t))\\ &= -r\tilde{X}(t)\mathrm{d}t + \gamma(t)\,\mathrm{d}\tilde{S}(t) + r\gamma(t)\tilde{S}(t)\,\mathrm{d}t + r\beta(t)\tilde{B}(t)\,\mathrm{d}t = \gamma(t)\,\mathrm{d}\tilde{S}(t).\end{aligned}$$

The process $\tilde{S}(t)$ is a martingale under the probability measure $\boldsymbol{P}_*$. Hence $\tilde{X}(0) = \boldsymbol{E}_* f(S(T))$ and $X(0) = \mathrm{e}^{-rT}\boldsymbol{E}_* f(S(T))$.

Let $X(0) = \mathrm{e}^{-rT}\boldsymbol{E} f(S_*(T))$. We need to show that the strategy defined in the lemma does exist and is admissible.

Consider the following Cauchy problem:

$$\frac{\partial V}{\partial t}(y,t) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 V}{\partial y_i\,\partial y_j}(y,t)\sum_{k=1}^{n}\sigma_{ik}\,\sigma_{jk} - \sum_{i=1}^{n}\frac{\partial V}{\partial y_i}(y,t)\sum_{k=1}^{n}\sigma_{ik} = 0, \tag{A.1}$$

$$V(y,T) = f(\mathrm{e}^{y_1},\dots,\mathrm{e}^{y_n}), \quad y = (y_1,\dots,y_n)\in\boldsymbol{R}^n. \tag{A.2}$$

First, we assume that $f(x)$ has a finite support inside the open domain $\mathring{\boldsymbol{R}}^n_+$, and $f(x)$ is smooth enough. Then the problem (A.1) and (A.2) has a classical solution. It can be seen that $H(x,t) \triangleq V(\ln x_1, \dots, \ln x_n, t)$ is a classical solution of (3.1). Applying Ito's formula, one has

$$\begin{aligned}\mathrm{d}X(t) &= r \times (t)\mathrm{d}t + \mathrm{e}^{r(t-T)}\\ &\quad\times\left(\frac{\partial H}{\partial t}(\tilde{S}(t),t) + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{n}(\sigma_{ik}\,\sigma_{jk})\tilde{S}_i(t)\tilde{S}_j(t)\frac{\partial^2 H}{\partial x_i\,\partial x_j}(S(t),t)\right.\\ &\quad\left.-r\sum_{i=1}^{n}\frac{\partial H}{\partial x_i}(\tilde{S}(t),t)\hat{S}_i(t)\right)\mathrm{d}t + \mathrm{e}^{r(t-T)}\sum_{i=1}^{n}\frac{\partial H}{\partial x_i}(\tilde{S}(t),t)\,\mathrm{d}\tilde{S}_i(t)\\ &= \gamma(t)'\,\mathrm{d}S(t) + \beta(t)\,\mathrm{d}B(t).\end{aligned} \tag{A.3}$$

Furthermore, $X(T) = f(S(T))$. Denote $Z(t) \triangleq \sigma'\boldsymbol{S}(t)\gamma(t)$. For any $\alpha\in\mathcal{A}$, consider the conditional probability space given that $a = \alpha$. By (A.3), we have in this space that

$$\mathrm{d}X(t) = Z(t)'\,\mathrm{d}w(t) + Z(t)'\sigma^{-1}\alpha\,\mathrm{d}t + r(X(t) - Z(t)'\sigma^{-1}\vec{e})B(t)\,\mathrm{d}t, \tag{A.4}$$

$$X(T) = f(S(T)). \tag{A.5}$$

The solution of the Eqs. (A.4) and (A.5), which constitute a backward stochastic differential equation, is a square integrable process pair $(X(\cdot), Z(\cdot))$, and there exists a constant c_0 such that

$$\sup_t \boldsymbol{E}\left\{|\,X(t)|^2\middle|\,a=\alpha\right\} + \boldsymbol{E}\left\{\int_0^T |Z(t)|^2\,\mathrm{d}t\middle|\,a=\alpha\right\} \le c_0\boldsymbol{E}\left\{f(S(T))^2|a=\alpha\right\}$$

for all $f(\cdot)$ and all non-random $\alpha\in\mathcal{A}$ (see, e.g. Yong and Zhou (1999), Chapter 7, Theorem 2.2). Hence

$$\sup_t \boldsymbol{E}|X(t)|^2 + \boldsymbol{E}\int_0^T |Z(t)|^2\,\mathrm{d}t \le c_0\boldsymbol{E}\,f(S(T))^2. \tag{A.6}$$

Now, let $f(x)$ be just a measurable function as given in the lemma. Let $f^{(i)}(x)$, $i = 1, 2, \ldots,$ be sufficiently smooth functions having finite supporters inside the open domain $\mathring{\boldsymbol{R}}^n_+$ such that

$$\boldsymbol{E}|f^{(i)}(S(T)) - f(S(T))|^2 \to 0 \quad \text{as} \quad i \to \infty.$$

Let $X^{(i)}(\cdot)$, $\gamma_k^{(i)}(\cdot)$, $\beta^{(i)}(\cdot)$, $H^{(i)}(\cdot)$ be the corresponding processes and functions. Then there exists a solution $H(\cdot)$ of (3.1) as a limit of $H^{(i)}(\cdot)$ in $\mathcal{Y}^1$

By (A.6) and the linearity of (A.4), we have

$$\begin{aligned}&\sup_t \boldsymbol{E}|X^{(i)}(t) - X^{(j)}(t)|^2 + \boldsymbol{E}\sum_{k=1}^{n}\int_0^T |S_k(t)(\gamma_k^{(i)}(t) - \gamma_k^{(j)}(t))|^2\,\mathrm{d}t \\ &\quad \le c_0 \boldsymbol{E}|f^{(i)}(S(T)) - f^{(j)}(S(T))|^2 \to 0 \quad \text{as} \quad i \to \infty.\end{aligned}$$

Hence $\{X^{(i)}(\cdot)\}$, $\{S_k(\cdot)\gamma_k^{(i)}(\cdot)\}$ and $\{\beta^{(i)}(\cdot)\}$ are Cauchy sequences in the space of square integrable processes, and have the corresponding limits $X(\cdot)$, $S_k(\cdot)\gamma_k(\cdot)$ and $\beta(\cdot)$ that are square integrable processes. This completes the proof. □

Now we turn to the proof of Theorem 3.1. To do this we need a series of lemmas.

For an $\alpha \in \mathcal{A}$, introduce the following random number

$$z(\alpha) \triangleq \exp\left(\int_0^T (\sigma^{-1}(\alpha - \vec{r}))'\,\mathrm{d}w(t) - \frac{1}{2}\int_0^T |\sigma^{-1}(\alpha - \vec{r})|^2\,\mathrm{d}t\right).$$

Furthermore, let

$$\mathcal{Z} \triangleq \int_{\mathcal{A}} \mathrm{d}\nu(\alpha) z(\alpha).$$

Lemma A.1. *The following holds:*

$$\mathcal{Z} = \frac{p(S_*(T), T)}{p_*(S_*(T), T)}. \tag{A.7}$$

Proof. First of all, it is easily seen that $z(\alpha) = \phi_1(\alpha, W(T))$, where $\phi_1 : \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}$ is a measurable function. Also, $W(T) = \sigma^{-1}v$, where $v = (v_1, \ldots, v_n)'$, $v_i = \ln(S_{*i}(T))/S_i(0)) - rT + T\sum_{j=1}^n \sigma_{ij}^2/2$. Hence $z(\alpha) = \phi_2(\alpha, S_*(T))$, where $\phi_2 : \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}$ is measurable. This leads to $\mathcal{Z} = \psi_0(S_*(T))$ for some measurable $\psi_0 : \boldsymbol{R}^n \to \boldsymbol{R}$.

Now,

$$\mathrm{d}S(t) = \boldsymbol{S}(t)a\,\mathrm{d}t + \boldsymbol{S}(t)\sigma\,\mathrm{d}w(t), \qquad \mathrm{d}S_*(t) = \boldsymbol{S}_*(t)\vec{r}\,\mathrm{d}t + \boldsymbol{S}_*(t)\sigma\,\mathrm{d}w(t).$$

By the Girsanov theorem, for any measurable bounded function $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$, we have

$$\int_{\boldsymbol{R}_+^n} p(x,T)\phi(x)\,\mathrm{d}x = \int_{\mathcal{A}} \mathrm{d}\nu(\alpha)\boldsymbol{E}\{\phi(S(T))|a$$
$$= \alpha\} = \int_{\mathcal{A}} \mathrm{d}\nu(\alpha)\int_{\boldsymbol{R}_+^n} \boldsymbol{E}\,\{z(\alpha)\phi(S_*(T))|\,a=\alpha, S_*(T)=x\}\,p_*(x,T)\,\mathrm{d}x$$
$$= \int_{\mathcal{A}} \mathrm{d}\nu(\alpha)\int_{\boldsymbol{R}_+^n} \boldsymbol{E}\,\{z(\alpha)|\,S_*(T)=x\}\,\phi(x)p_*(x,T)\,\mathrm{d}x$$
$$= \int_{\boldsymbol{R}_+^n} \psi_0(x)\phi(x)p_*(x,T)\,\mathrm{d}x.$$

It follows

$$\psi_0(x) = \psi(x) = \frac{p(x,T)}{p_*(x,T)}, \quad \forall x. \tag{A.8}$$

This completes the Proof of Lemma A.1. □

Lemma A.2. *For* $k=0,1$, *we have* $\sup_{\alpha\in\mathcal{A}}\boldsymbol{E}z(\alpha)^k\mathcal{Z}^c < +\infty,\ \forall c\in\boldsymbol{R}$.

Proof. First of all, it is clear that

$$\sup_{\alpha\in\mathcal{A}}\boldsymbol{E}z(\alpha)^c < +\infty, \quad \forall c\in\boldsymbol{R},$$

and

$$\sup_{\alpha\in\mathcal{A}}\boldsymbol{E}z(\alpha)^k\mathcal{Z}^c = \sup_{\alpha\in\mathcal{A}}\boldsymbol{E}z(\alpha)^k\left(\int_{\mathcal{A}}\mathrm{d}\nu(\alpha_1)z(\alpha_1)\right)^c < +\infty, \quad \forall\, c>0.$$

Now, for any $c<0$, the function y^c is convex over $y>0$. Hence

$$\begin{aligned}\sup_{\alpha\in\mathcal{A}}\boldsymbol{E}z(\alpha)^k\mathcal{Z}^c &= \sup_{\alpha\in\mathcal{A}}\boldsymbol{E}z(\alpha)^k\left(\int_{\mathcal{A}}\mathrm{d}\nu(\alpha_1)z(\alpha_1)\right)^c\\ &\le \sup_{\alpha\in\mathcal{A}}\boldsymbol{E}z(\alpha)^k\int_{\mathcal{A}}\mathrm{d}\nu(\alpha_1)z(\alpha_1)^c < +\infty.\end{aligned}$$

This completes the proof of Lemma A.2. □

To proceed, consider a class of random numbers (claims) $\bar{\Phi}$ that are $\mathcal{F}_T^S$-measurable. By definition, for a $\xi\in\bar{\Phi}$, there exists a measurable function $\phi : C([0,T];\boldsymbol{R}^n)\to\boldsymbol{R}$ such that $\xi=\phi(S(\cdot))$. We denote $\xi_*\triangleq\phi(S_*(\cdot))$, corresponding to each $\xi\in\bar{\Phi}$.

Let Φ be a subset of $\bar{\Phi}$ consisting of all claims $\xi\in\bar{\Phi}$ such that $E|\xi_*|^2<+\infty$ and $\boldsymbol{E}U^+(\xi)<+\infty$, where $U^+(x)\triangleq\max(U(x),0)$. Further, for any $\lambda>q_0$, define the random number

$$\eta(\lambda)\triangleq f(\lambda, S(T)), \tag{A.9}$$

where $f(\cdot)$ is defined by (3.2).

Now, introduce the following functionals $J_i : \Phi \to \boldsymbol{R},\ i = 0, 1$:

$$J_0(\xi) \triangleq \boldsymbol{E}U(\xi), \qquad j_1(\xi) \triangleq \boldsymbol{E}\xi_* - \mathrm{e}^{rT}X(0),$$

and set

$$\Phi_0 \triangleq \{\xi \in \Phi : J_1(\xi) = 0, \qquad h_1(S(T)) \le \xi \le h_2(S(T)), \quad \text{a.s.}\}.$$

Consider an auxiliary optimization problem

$$\text{Maximize } J_0(\xi) \text{ over class } \Phi_0. \tag{A.10}$$

Lemma A.3. *The optimization problem* (A.10) *has an optimal solution* $\hat{\xi} \triangleq \eta(\hat{\lambda}) \in \Phi_0$ *where* $\hat{\lambda} \in (q_0, +\infty)\backslash\{0\}$ *is a number such that*

$$\boldsymbol{E}\hat{\xi}_* = \mathrm{e}^{rT}X(0), \tag{A.11}$$

namely, (3.4) *holds. Moreover, if* (3.7) *holds, then* $\hat{\xi}$ *is determined uniquely:* $\eta(\hat{\lambda}_1) = \eta(\hat{\lambda}_2)$ *a.s. whenever* $\hat{\lambda}_1$ *and* $\hat{\lambda}_2$ *satisfy* (A.11).

Proof. Let $\xi \in \Phi$ with $\xi = \bar{\phi}(S(\cdot))$, where $\bar{\phi} : C([0,T]; \boldsymbol{R}^n) \to \boldsymbol{R}$ is a measurable function. By the Girsanov theorem, we have

$$\begin{aligned} J_0(\xi) &= \boldsymbol{E}U(\xi) = \int_{\mathcal{A}} \mathrm{d}\nu(\alpha)\boldsymbol{E}\{U(\xi)|a = \alpha\} \\ &= \int_{\mathcal{A}} \mathrm{d}\nu(\alpha)\boldsymbol{E}\{z(\alpha)U(\xi_*)|a = \alpha\} = \boldsymbol{E}\mathcal{Z}U(\xi_*). \end{aligned}$$

Introduce a function $L(\xi, \lambda) \triangleq J_0(\xi) - \lambda J_1(\xi)$, where $\xi \in \bar{\Phi}, \lambda \ge q_0, \lambda > -\infty$. By Lemma A.l, we have $\mathcal{Z} = \psi(S_*(T))$, and

$$L(\xi, \lambda) = \boldsymbol{E}(\psi(S_*(T))U(\xi_*) - \lambda\xi_*) + \lambda\,\mathrm{e}^{rT}X(0). \tag{A.12}$$

By definition, $\eta_*(\lambda) = f(\lambda, S_*(T))$. In view of Assumption 2.2, for any $\omega \in \Omega$ and $\lambda > q_0$ such that $\lambda/\psi(S_*(T)) \in I$, the random number $\eta_*(\lambda)$ provides the maximum of the function $\psi(S_*(T))U(\xi_*) - \lambda\xi_*$ over the interval $(h_1(S_*(T)), h_2(S_*(T)))$. It then follows from (3.3) that $\boldsymbol{P}(\lambda/\psi(S_*(T)) \in I) = 1$. By Lemma A.2 and Assumption 2.2,

$$\begin{aligned} |f(\lambda, S_*(T))| &\le c_1 \left(\left| \frac{\psi(S_*(T))}{\lambda} \right|^c + \left| \frac{\psi(S_*(T))}{\lambda} \right|^{-c} + 1 \right) \\ &= c_1 \left(\left| \frac{\mathcal{Z}}{\lambda} \right|^c + \left| \frac{\mathcal{Z}}{\lambda} \right|^{-c} + 1 \right), \quad \text{a.s..} \end{aligned}$$

By Lemma A.2, $\boldsymbol{E}|f(\lambda, S_*(T))|^c < +\infty (\forall\, \lambda \ne 0,\ \forall\, c > 0)$, and

$$\boldsymbol{E}|f(\lambda, S(T))|^c = \int_{\mathcal{A}} \mathrm{d}\nu(\alpha)\boldsymbol{E}\{z(\alpha)|f(\lambda, S_*(T))|^c a = \alpha\} < +\infty, \quad \forall\lambda \ne 0, \forall c > 0.$$

This yields $\boldsymbol{E}|\hat{\xi}_*|^c < +\infty, \boldsymbol{E}|\hat{\xi}|^c < +\infty$ $(\forall c > 0)$. It follows that $\boldsymbol{E}U^+(\hat{\xi}) < +\infty$. Thus, $\hat{\xi}(\hat{\lambda}) \in \Phi$ and, by definition, $\hat{\xi}(\hat{\lambda}) \in \Phi_0$. Now,

$$L(\xi, \hat{\lambda}) \leq L(\hat{\xi}(\hat{\lambda}), \hat{\lambda}), \quad \forall \xi \in \Phi_0. \tag{A.13}$$

Let $\xi \in \Phi_0$ be arbitrary. Since $J_1(\hat{\xi}(\hat{\lambda})) = 0$, we have

$$\begin{aligned} J_0(\xi) - J_0(\hat{\xi}(\hat{\lambda})) &= J_0(\xi) - J_0(\hat{\xi}(\hat{\lambda})) - \hat{\lambda} J_1(\hat{\xi}(\hat{\lambda})) \\ &= J_0(\xi) + \hat{\lambda} J_1(\xi) - J_0(\hat{\xi}(\hat{\lambda})) - \hat{\lambda} J_1(\hat{\xi}(\hat{\lambda})) = L(\xi, \hat{\lambda}) - L(\hat{\xi}(\hat{\lambda}), \hat{\lambda}) \leq 0. \end{aligned}$$

Consequently, $\hat{\xi}(\hat{\lambda})$ is an optimal solution of the problem (A.10).

To prove the uniqueness of solution under (3.7), let $\xi' \in \Phi_0$ be an optimal solution of the problem (A.10) and $\hat{\lambda}$ be any number such that (3.4) holds. Then,

$$L(\xi', \hat{\lambda}) = J_0(\xi') \geq J_0(\hat{\xi}) = L(\hat{\xi}, \hat{\lambda}).$$

By Assumption 2.2, for any $\omega \in \Omega$ and $\lambda > q_0$ such that $\lambda/\psi(S_*(T)) \in I_0$, there exists a unique random number achieving the maximum of the function $\psi(S_*(T))U(\xi_*) - \lambda\xi_*$ over the interval $(h_1(S_*(T)), h_2(S_*(T)))$. Thus, $\xi' = \eta(\hat{\lambda})$ a.s.. This completes the proof. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. It has been shown in the Proof of Lemma A.3 that $\boldsymbol{E}|f(\lambda, S_*(T))|^2 < +\infty$ and $\boldsymbol{E}|f(\lambda, S(T))|^2 < +\infty \quad \forall \lambda \neq 0$. Moreover, it follows from Lemma 3.1 that the Eq. (3.1) admits a solution for any terminal condition $f = f(\lambda, x)$ with $\lambda \neq 0$. Furthermore, it follows from Lemma A.3 that $\hat{\xi} \triangleq f(\hat{\lambda}, S(T)) \in \Phi_0$. The self-financing strategy that replicates $\hat{\xi}$ is uniquely defined by $f(\hat{\lambda}, x)$ as stipulated in Lemma 3.1. In addition by Lemma A.3, it is an optimal strategy, which is unique if $\hat{\xi}$ is unique. This completes the proof of Theorem 3.1. □

Proof of Lemma 3.2. By the definition of $f(\cdot)$, we have $h_1(x) \leq f(\lambda, x) \leq h_2(x)$. Moreover, $\boldsymbol{E}|h_i(S_*(T))| < +\infty$. It then follows from Dominate Convergence Theorem that either

$$\begin{aligned} &\boldsymbol{E} f(\lambda, S_*(T)) \to \boldsymbol{E} h_1(S_*(T)) \quad \text{as } \lambda \to +\infty, \\ &\boldsymbol{E} f(\lambda, S_*(T)) \to \boldsymbol{E} h_2(S_*(T)) \quad \text{as } \lambda \to q_0, \end{aligned}$$

or

$$\begin{aligned} &\boldsymbol{E} f(\lambda, S_*(T)) \to \boldsymbol{E} h_2(S_*(T)) \quad \text{as } \lambda \to +\infty, \\ &\boldsymbol{E} f(\lambda, S_*(T)) \to \boldsymbol{E} h_1(S_*(T)) \quad \text{as } \lambda \to q_0. \end{aligned}$$

Hence by Assumption 2.1 there exists $\hat{\lambda} > q_0$ such that (3.4) holds. □

Proof of Thorem 3.2. For any fixed $a \in \mathcal{A}$, let $p_a^{(i)}(x_i, t)$ be the probability density function for the stock price $S_i(t)$. Furthermore, let $p_*^{(i)}(x_i, t)$ be the probability density function for the stock price $S_i(t)$ with $a_i(\cdot) \equiv r$. It can be easily seen that $S_i(T) = S_i(0)\exp\{\mu_i + rT - \bar{\sigma}_i^2/2 + \xi_i\}$, where $\mu_i, \bar{\sigma}_i$ are defined in (3.9), and, ξ_i, $i = 1, 2, \cdots$, are independent Gaussian random numbers such that $\boldsymbol{E}\xi_i = 0$, $\boldsymbol{E}\xi_i^2 = \bar{\sigma}_i^2$. Hence

$$p_*^{(i)}(x_i, T) = \frac{1}{x_i\sqrt{2\pi}\bar{\sigma}_i} \exp\frac{-(\ln(x_i) - \ln(S_i(0)) - rT + \bar{\sigma}_i^2/2)^2}{2\bar{\sigma}_i^2}, \tag{A.14}$$

$$p_a^{(i)}(x_i, T) = \frac{1}{x_i\sqrt{2\pi}\bar{\sigma}_i} \exp\frac{-(\ln(x_i) - \ln(S_i(0)) - \mu_i - rT + \bar{\sigma}_i^2/2)^2}{2\bar{\sigma}_i^2}. \tag{A.15}$$

Denote

$$\varepsilon_i \triangleq \ln(x_i) - \ln(S_i(0)) - rT + \frac{\bar{\sigma}_i^2}{2}.$$

We have

$$\begin{aligned}\exp\frac{-((\varepsilon_i - \mu_i)^2 - \varepsilon_i^2)}{2\bar{\sigma}_i^2} &= \exp\frac{\varepsilon_i\mu_i}{\bar{\sigma}_i^2}\exp\frac{-\mu_i^2}{2\bar{\sigma}_i^2}\\ &= \left(\frac{x_i}{S_i(0)}\right)^{\mu_i/\bar{\sigma}_i^2}\exp\frac{\mu_i(\bar{\sigma}_i^2 - 2rT) - \mu_i^2}{2\bar{\sigma}_i^2}.\end{aligned}$$

It then follows that

$$\frac{p_a^{(i)}(x_i, T)}{p_*^{(i)}(x_i, T)} = \left(\frac{x_i}{S_i(0)}\right)^{\mu_i/\bar{\sigma}_i^2} e^{\theta_i}, \tag{A.16}$$

where θ_i is defined in (3.9). Furthermore, we have

$$p_a(x, t) = \prod_{i=1}^{n} p_a^{(i)}(x_i, t), \qquad p_*(x, t) = \prod_{i=1}^{n} p_*^{(i)}(x_i, t), \quad x = (x_1, \dots, x_n). \tag{A.17}$$

From (A.15)–(A.17), we obtain (3.11). The proof is completed. □

Proof of Theorem 4.1. Let G' be the set of admissible $\gamma(\cdot)$ such that the constraints in (2.6) hold. Denote by $J'(\gamma(\cdot))$ and $J''(\gamma(\cdot))$ the functionals to be maximized in (2.6) and (4.1), respectively.

Suppose $\gamma(\cdot)$ is an optimal strategy for the Problem (4.1) and (4.2). Construct the following strategy: $\hat{\gamma}(t) = \gamma(t)$ for $t \leq \tau$, and $\hat{\gamma}(t) = 0$ for $t > \tau$, where $\tau \triangleq \tau_1 \wedge \tau_2$. Clearly $\hat{\gamma}(\cdot) \in G'$ and $J'(\hat{\gamma}(\cdot)) = J''(\gamma(\cdot))$. Hence

$$\sup_{\gamma(\cdot)\in G'} J'(\gamma(\cdot)) \geq \sup_{\gamma(\cdot)} J''(\gamma(\cdot)).$$

On the other hand, let $\bar{\gamma}(\cdot)$ be the optimal strategy for the problem (2.6). This strategy is unique and is given by (3.8), (3.5), and (4.4). The corresponding optimal wealth process $X(t)$ is given by (3.6) and (4.5). It can be easily seen from these equations that $e^{r(T-t)}X(t) \in$

(k_1, k_2), $\forall t < T$, a.s., where $X(t)$ is the corresponding optimal wealth process. Hence $J'(\bar{\gamma}(\cdot)) = J''(\bar{\gamma}(\cdot))$, leading to

$$\sup_{\gamma(\cdot)\in G'} J'(\gamma(\cdot)) \le \sup_{\gamma(\cdot)} J''(\gamma(\cdot)).$$

This completes the proof. □

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