Parabolic equations in unbounded cylinders and estimates for distances between first exit times

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Abstract

We consider first exit times and their dependence on variations of parameters for diffusion processes. Estimates of L_p-distances and some other distances between two exit times are obtained via parabolic Kolmogorov's equations with infinite horizon.

Introduction

It is known that first exit times from a region for smooth solutions of ordinary equations do not depend continuously on variations of the initial data or the coefficients. However, first exit times for non-smooth trajectories of diffusion processes have some path-wise regularity with respect to these variations (some related results can be found in author's papers (1987),(1992)).

We study path-wise dependence on variations of initial data and coefficients for first exit times of diffusion processes from a domain $D \subset \mathbf{R}^n$. We present an effective estimate of distances between exit times via estimates for solutions $v(x,t)$ of backward Kolmogorov's parabolic equations in the unbounded cylinder $D \times [0,+\infty)$, when the Cauchy boundary condition is replaced by the condition $\sup_{t\geq 0}\|v(\cdot,t)\|_{L_2(D)} < +\infty$. These problems are sometimes called Fourier problems.

1 Estimates of distances between first exit times

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space. Consider two n-dimensional diffusion processes $y_i(t)$, $i = 1, 2$, such that

$$\begin{cases} dy_i(t) = f_i(y(t), t)dt + \beta_i(y_i(t), t)dw(t), \quad t > 0, \\ y(0) = a_i. \end{cases} \tag{1.1}$$

Here $w(t)$ is a standard n-dimensional Wiener process, $f_i : Q \to \mathbf{R}^n$ and $\beta_i : Q \to \mathbf{R}^{n\times n}$ are non-random functions, where $Q = D \times [0, +\infty)$, $D \in \mathbf{R}^n$ is a bounded domain. The random vectors a_i with values in $\bar{D}$ does not depend on $w(\cdot)$.

Set $b \triangleq \beta_i\beta_i^\top$.

We assume that

$$
\begin{aligned}
c_b &\triangleq \sup_{(x,t)\in Q,\ i=1,2} |b_i(x,t)| < +\infty,\\
c_f &\triangleq \sup_{(x,t)\in Q,\ i=1,2} |f_i(x,t)| < +\infty,\\
\bar{c}_b &\triangleq \operatorname{ess\,sup}_{(x,t)\in Q,\ i=1,2} |\tfrac{\partial b_i}{\partial x}(x,t)| < +\infty,\\
\delta &\triangleq \inf_{(x,t)\in Q,\ \xi\in\mathbf{R}^n,\ i=1,2} \tfrac{\xi^\top b_i(x,t)\xi}{|\xi|^2} > 0.
\end{aligned}
\tag{1.2}
$$

Set

$$
\mathcal{P}_0 \triangleq (n, \quad D, \quad c_\beta, \quad \bar{c}_b, \quad c_f, \quad \delta).
$$

Let $\tau_i \triangleq \inf\{t \geq 0 : \; y_i(t) \notin D\}$ and
$\widetilde{\tau} \triangleq \tau_1 \wedge \tau_2 = \min(\tau_1, \tau_2)$.

Theorem 1.1 *There exist $\lambda_0 > 0$ such that for any $\lambda < \lambda_0$ there exists $C(\lambda) = C(\lambda, \mathcal{P}_0) > 0$ such that*

$$\mathbf{E}\frac{1}{\lambda}\left[e^{\lambda|\tau_1-\tau_2|} - 1\right] \leq C(\lambda)\mathbf{E}|y_1(\widetilde{\tau}) - y_2(\widetilde{\tau})|. \tag{1.3}$$

Clearly, $|\tau_1 - \tau_2|^p \leq p!\, \lambda^{-p}\left[e^{\lambda|\tau_1-\tau_2|} - 1\right]$ for $p = 1, 2, \ldots$.

Corollary 1.1 *For any $p = 1, 2, ..,$, there exists $C = C(p, \mathcal{P}) > 0$ such that*

$$\mathbf{E}|\tau_1 - \tau_2|^p \leq C\mathbf{E}|y_1(\widetilde{\tau}) - y_2(\widetilde{\tau})|. \tag{1.4}$$

Example Let $n = 1$, $y_i(t) = a_i + w_t$, where a_i are non-random, $a_i \in D$, and $D \subset \mathbf{R}$ is a given interval. We have that

$$\mathbf{E}|y_1(\widehat{\tau}) - y_2(\widehat{\tau})| = |a_1 - a_2|.$$

Then it follows that

$$\frac{1}{\lambda}\mathbf{E}\{e^{\lambda|\tau_1-\tau_2|} - 1\} \leq C(\lambda)|a_1 - a_2|$$

and

$$\mathbf{E}|\tau_1 - \tau_2|^p \leq \widehat{c}(p)|a_1 - a_2|$$

for all $p \geq 1$.

2 Calculation of constants via parabolic equations in unbounded cylinders

Let $f(x,t) : Q \to \mathbf{R}^n$ and $\beta(x,t) : Q \to \mathbf{R}^{n\times n}$ be non-random uniformly continuous functions such that all the components of the functions f and β are continuously differentiable with respect x, $Q \triangleq D \times [0,+\infty)$.

Introduce the differential operator

$$\mathcal{A}(t)v(x) \triangleq \frac{\partial v}{\partial x}(x)f(x,t) + \frac{1}{2}\sum_{k,m=1}^{n} b^{(km)}(x,t)\frac{\partial^2 v}{\partial x^{k)}\partial x^{(m)}}(x,t). \tag{2.1}$$

Here $x^{(k)}$ and $b^{(km)}$ are the components of vector x and the matrix $b \triangleq \beta\beta^{\top}$.

We assume that

$$
\begin{aligned}
&c_b \triangleq \sup_{(x,t)\in Q} |b(x,t)| < +\infty,\\
&c_f \triangleq \sup_{(x,t)\in Q} |f(x,t)| < +\infty,\\
&\bar{c}_b \triangleq \operatorname{ess\,sup}_{(x,t)\in Q} |\tfrac{\partial b}{\partial x}(x,t)| < +\infty,\\
&\delta \triangleq \inf_{(x,t)\in Q,\ \xi\in \mathbf{R}^n} \tfrac{\xi^\top b(x,t)\xi}{|\xi|^2} > 0.
\end{aligned}
\tag{2.2}
$$

Set

$$\mathcal{P} \triangleq (n, \quad D, \quad c_\beta, \quad \bar{c}_b, \quad c_f, \quad \delta).$$

Let $q(x,t) : Q \to \mathbf{R}$ be a measurable function.

Consider the boundary value problem in the semi-infinite cylinder Q

$$\begin{cases} \frac{\partial v}{\partial t}(x,t) + \mathcal{A}(t)v(x,t) + q(x,t)v(x,t) = -\varphi(x,t), \\ v(x,t)|_{x\in\partial D} = 0, \\ \operatorname{ess\,sup}_{s\geq 0} \|v(\cdot,t)\|_{L_2(D)} < +\infty. \end{cases} \tag{2.3}$$

Let

$$\mathcal{P}_q \triangleq \left(\mathcal{P}_0, \sup_{(x,t)\in Q} q(x,t)\right), \qquad \mathcal{P}_{|q|} \triangleq \left(\mathcal{P}_0, \sup_{(x,t)\in Q} |q(x,t)|\right).$$

Some definitions

Let $D \subset \mathbf{R}^n$ be a bounded domain with C^1-smooth boundary ∂D.

Let $H^0 \triangleq L_2(D)$, and let $H^1 \triangleq \overset{0}{W}{}_2^1(D)$. Let H^{-1} be the dual space to H^1.

Let ℓ_m denotes the Lebesgue measure in $\mathbf{R}^m$, and let $\overline{\mathcal{B}}_m$ be the Lebesgue σ-algebra. Let

$$\mathcal{C}^0(0,T) \triangleq C([0,T]; H^0),$$

$$X_r^k(0,T) \triangleq L^r([0,T], \overline{\mathcal{B}}_1, \ell_1; H^k), \quad k = 0, \pm 1.$$

Introduce spaces $Y^k(s,+\infty)$ of functions $u : [s,+\infty) \to H^k$ with finite norm

$$\|u\|_{Y^k(s,+\infty)} = \sup_{i=0,1,2\ldots} \left(\int_{s+i}^{s+i+1} \|u(\cdot,t)\|_{H^k}^2 dt\right)^{1/2}.$$

Let $X_{r,loc}^k(0,\infty)$ be the set of functions $u(x,t) : D \times (s_1, s_2) \to \mathbf{R}$ such that $u|_{D\times[s,t]} \in X_r^k(s,t)$ for all s, t.

For $\gamma \geq 1$, let $W_\gamma^{2,1}(\mathcal{Q})$ be a Banach space of functions $u : \mathcal{Q} \to \mathbf{R}$ that belong to $L_\gamma(\mathcal{Q})$ together with all derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x_k}$, $\frac{\partial^2 u}{\partial x_k \partial x_m}$, $k, m = 1, ..., n$, with finite norm

$$\|u\|_{W_\gamma^{2,1}(\mathcal{Q})} \triangleq \|u\|_{L_\gamma(\mathcal{Q})} + \left\|\frac{\partial u}{\partial t}\right\|_{L_\gamma(\mathcal{Q})} + \sum_{k=1}^n \left\|\frac{\partial u}{\partial x_k}\right\|_{L_\gamma(\mathcal{Q})} + \sum_{k,m=1}^n \left\|\frac{\partial^2 u}{\partial x_k \partial x_m}\right\|_{L_\gamma(\mathcal{Q})}.$$

Let $\alpha \in (0, 1)$ be a non-integer number. We will say that a function $u : \mathcal{Q} \to \mathbf{R}$ belongs to the class $\mathcal{H}^{1+\alpha,(1+\alpha)/2}(\mathcal{Q})$ if u and $\partial u/\partial x$ are continuous, and

$$\langle\langle u \rangle\rangle_{\mathcal{Q}}^{(1+\alpha)} \triangleq \langle u \rangle_{\mathcal{Q}}^{(\alpha)} + \sum_{k=1}^n \left\langle \frac{\partial u}{\partial x_k} \right\rangle_{\mathcal{Q}}^{(\alpha)} < +\infty,$$

where

$$\langle u \rangle_{\mathcal{Q}}^{(\alpha)} \triangleq \max_{(x,t)\in\mathcal{Q}} |u(x,t)| + \sup_{x,x'\in D,\ t\in[s,T]} \frac{|u(x,t) - u(x',t)|}{|x - x'|^\alpha} + \sup_{x\in D,\ t,t'\in[s,T]} \frac{|u(x,t) - u(x,t')|}{|t - t'|^{\alpha/2}}.$$

This class is a special case of the Hölder space from Ladyzhenskaya *et al* (1968)

Theorem 2.1 *There exists $\lambda > 0$ such that if $q(x,t) \le \lambda$ for all (x,t), then*

(i) For any $\varphi \in Y^{-1}(0,+\infty)$ there exists the unique (up to equivalency) solution $v : D \times (0,+\infty) \to \mathbf{R}$ of the problem (2.3) in the class $X^0_\infty(0,+\infty) \cap X^1_{2,loc}(0,+\infty)$;

(ii) $v \in \mathcal{C}^0(0,+\infty)$, and there exists a constant $c = c(\mathcal{P},\lambda)$ such that

$$\sup_{t\in[0,+\infty)} \|v(\cdot,t)\|_{H^0} + \|v\|_{Y^1(0,+\infty)} \le c\|\varphi\|_{Y^{-1}(0,+\infty)}. \tag{2.4}$$

Notice that the parabolic equation in (2.3) is in the sense of Sobolev generalized functions, and we assume that the boundary condition $v(x,t)|_{x\in\partial D} = 0$ is satisfied if $v(\cdot,t) \in H^1 = \overset{0}{W}{}^1_2(D)$ for a.e. t. Definitions of spaces ensure that the statement of the problem problem (2.3) has sense.

Theorem 2.2 *Let $q(x,t) \le \lambda$ for all (x,t), where λ is such as in Theorem 2.1. Then*

(i) If $\sup_{t\ge 0} \|\varphi|_{Q_t}\|_{L_\gamma(Q_t)} < +\infty$ for $\gamma \ge 2$, then $v|_{Q_t} \in W^{2,1}_\gamma(Q_t)$ for all $t \ge 0$, and there exists a constant $c = c(\mathcal{P}_{|q|}, \gamma)$ such that

$$\sup_{t\ge 0} \|v|_{Q_t}\|_{W^{2,1}_\gamma(Q_t)} \le c \sup_{t\ge 0} \|\varphi\|_{L_\gamma(Q_t)}. \tag{2.5}$$

(ii) If $\sup_{t\ge 0} \|\varphi|_{Q_t}\|_{L_\gamma(Q_t)} < +\infty$ for $\gamma > n+2$, then the function $v(x,t)$ and its derivatives $\partial v(x,t)/\partial x_k$ are continuous, bounded, and belong to the Hölder class $\mathcal{H}^{1+\alpha,(1+\alpha)/2}(Q_t)$ for all $t \ge 0$, where $\alpha \triangleq 1 - (n+2)/\gamma$, $k = 1, ..., n$. Moreover, there exists a constant $c = c(\mathcal{P}_{|q|}, \gamma)$ such that

$$\sup_{t\ge 0} \langle\langle u|_{Q_t}\rangle\rangle^{(1+\alpha)}_{Q_t} \le c \sup_{t\ge 0} \|\varphi\|_{L_\gamma(Q_t)}. \tag{2.6}$$

2.1 Probabilistic interpretation and Krylov's estimates

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space. Let n-dimensional diffusion processes $y(t)$ be solution of the Itô's equation

$$\begin{cases} dy(t) = f(y(t), t)dt + \beta(y(t), t)dw(t), \quad t > s, \\ y(s) = a. \end{cases} \tag{2.7}$$

Here $w(t)$ is a standard n-dimensional Wiener process. The random vectors a with values in $\bar{D}$ does not depend on $w(t) - w(r)$ for all $t > r > s$.

We denote by $y^{a,s}(t)$ the solution of (2.7). Let $\tau^{a,s} \triangleq \inf\{t : \; y^{a,s}(t) \notin D\}$.

Theorem 2.3 *If $\varphi \in Y^0(0,+\infty)$ is a Borel measurable function then*

$$v(x,s) = \mathbf{E}\int_s^{\tau^{x,s}} \varphi(y^{x,s}(t),t)\exp\Big(\int_s^t q(y^{x,s}(r),r)dr\Big)dt, \tag{2.8}$$

and this equality holds for all $s \geq 0$ for a.e. $x \in D$;

(ii) If $\varphi(x,t)$ is uniformly Hölder and bounded then the function $v(x,t)$ and its derivative $\partial v(x,t)/\partial x$ are continuous and uniformly bounded.

Remark 2.1 *The Krylov's estimates give estimation of $\sup_{x\in D}|v(x,s)|$ for $\lambda \leq 0$ via the norm of φ in $L_{n+1}(D\times(s,+\infty))$ or via $\|\varphi\|_{L_n(D)}$ for independent on t functions $\varphi(x,t)=\varphi(x)$ (see Theorem* II.*4.2 from Krylov (1980)) .*

2.2 Quasi-periodic conditions

Consider now the boundary value problem with quasi-periodical conditions in the cylinder $D \times [0,1]$

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + \mathcal{A}(t)u(x,t) + q(x,t)u(x,t) = -\varphi(x,t), \\ u(x,t)|_{x\in\partial D} = 0, \\ \mu u(x,0) - u(x,1) = 0. \end{cases} \tag{2.9}$$

Theorem 2.4 *Let $\mu \neq 0$. There exists $\lambda > 0$ such that if $q(x,t)+\ln|\mu| \leq \lambda$ for all (x,t), then the following holds.*

(i) For any $\varphi \in X_2^{-1}(0,1)$ there exists the unique (up to equivalency) solution $u : D \times (0,1) \to \mathbf{R}$ of the problem (2.9) in the class $\mathcal{C}^0(0,1) \cap X_2^1(0,1)$;

(ii) there exists a constant $c = c(\mathcal{P},\mu,\lambda)$ such that

$$\sup_{t\in([0,1]} \|u(\cdot,t)\|_{H^0} + \|u\|_{X_2^1(0,1)} \leq c\|\varphi\|_{X_2^{-1}(0,1)}; \tag{2.10}$$

(iii) If $\varphi(x,t)$ is uniformly Hölder and bounded then the functions $u(x,t)$ and $\partial u(x,t)/\partial x$ are continuous and uniformly bounded.

2.3 Constants for estimates of distances between first exit times

In fact, the constant $C(\lambda)$ is

$$C(\lambda) = \max_{i=1,2} \sup_{(x,t)\in Q} \left| \frac{dv_i}{dx}(x,t) \right| \tag{2.11}$$

where v_i is the the boundary value problem in Q for $i = 1, 2$

$$\begin{cases} \frac{\partial v_i}{\partial t}(x,t) + \mathcal{A}_i(t) v_i(x,t) + \lambda v(x,t) = -1, \\ v_i(x,t)|_{x\in\partial D} = 0, \\ \operatorname{ess\,sup}_{t>0} \|v_i(\cdot,t)\|_{L_2(D)} < +\infty, \end{cases} \tag{2.12}$$

where

$$\mathcal{A}_i(t)v(x) \triangleq \frac{\partial v}{\partial x}(x) f_i(x,t) + \frac{1}{2} \sum_{k,m=1}^{n} b_i^{(km)}(x,t) \frac{\partial^2 v}{\partial x^{k)} \partial x^{(m)}}(x,t). \tag{2.13}$$

Here $b_i^{(km)}$ are the components of the matrices $b_i \triangleq \beta_i \beta_i^\top$.

Remark 2.2 *We have assumed that the boundaries and coefficients are smooth enough, the diffusion is non-degenerate, and the domain D is bounded. In fact, we need these conditions only to ensure that the right hand part of (2.11) is finite.*

3 How to find the upper bound λ

Lemma 3.1 • *Under assumptions of Theorem 2.1, there exists $\nu \in (0,1)$ such that $\nu = \nu(\mathcal{P})$ depends only on $\mathcal{P}$ and*

$$\mathbf{P}(\tau^{a,s} > s+1) \le \nu$$

for any $s > 0$ and for any random vector a such that $a \in D$ a.s., a does not depend on $w(t) - w(r)$ for all $t > r > s$.

- *One can take*

$$\lambda = -\ln \nu.$$

4 Proofs

Proof of Lemma 3.1. Let $\mathcal{M} \triangleq \Theta(\mathcal{P}_0)\times D\times[0,+\infty)$. Let $\mu=(f,\beta,x,s)\in\mathcal{M}$ be given.

Clearly, there exists a finite interval $D_1 \triangleq (d_1,d_2)\subset \mathbf{R}$ and a bounded domain $D_{n-1}\subset\mathbf{R}^{n-1}$ such that $D\subset D_1\times D_{n-1}$.

Let $\tau_1^{x,s}\triangleq\inf\{t\ge s:\ y_1^{x,s}(t)\notin D_1\}$, where $y_1^{x,s}(t)$ is the first component of the vector $y^{x,s}(t)=(y_1^{x,s}(t),...,y_n^{x,s}(t))$. We have that

$$\mathbf{P}(\tau^{x,s}>s+1)\le\mathbf{P}(\tau_1^{x,s}>s+1)=\mathbf{P}(y_1^{x,s}(t)\in D_1\ \forall t\in[s,s+1]). \quad (4.1)$$

Let $M^\mu(t)\triangleq\int_s^t\beta_1(y^{x,s}(r),r)dw(r)$, $t\ge s$, where β_1 is the first row of the matrix β. Let $\widehat{D}_1\triangleq(d_1+K_1,d_2+K_2)$, where $K_1\triangleq -d_2-\sup_{x,t}|f_1(x,t)|$, $K_2\triangleq -d_1+\sup_{x,t}|f_1(x,t)|$. Clearly, $\widehat{D}_1$ depends only on n,D, and c_f. It is easy to see that

$$\mathbf{P}(y_1^{x,s}(t)\in D_1\ \forall t\in[s,s+1])\le\mathbf{P}(M^\mu(t)\in\widehat{D}_1\ \forall t\in[s,s+1]). \quad (4.2)$$

Further,

$$\beta_1(y^{x,s}(t),t)^\top\beta_1(y^{x,s}(t),t)=|\beta_1(y^{x,s}(t),t)|^2\in[\delta,c_\beta], \quad (4.3)$$

where δ and c_β are such as defined in (1.2). Clearly, $M^\mu(t)$ is a martingale vanishing at s with quadratic variation process

$$[M^\mu]_t\triangleq\int_s^t|\beta_1(y^{x,s}(r),r)|^2dr,\qquad t\ge s.$$

Let $\theta^\mu(t)\triangleq\inf\{r\ge s:\ [M^\mu]_r>t-s\}$. Note that $\theta^\mu(s)=s$, and the function $\theta^\mu(t)$ is strictly increasing in $t>s$ given (x,s). By Dambis–Dubins–Schwarz Theorem (see, e.g., Revuz and Yor (1999)), the process $B^\mu(t)\triangleq M(\theta^\mu(t))$ is a Brownian motion vanishing at s, i.e., $B^\mu(s)=0$, and $M^\mu(t)=B^\mu(s+[M^\mu]_t)$. Clearly,

$$\begin{aligned}\mathbf{P}(M^\mu(t)\in\widehat{D}_1\quad\forall t\in[s,s+1])&=\mathbf{P}(B^\mu(s+[M^\mu]_t)\in\widehat{D}_1\quad\forall t\in[s,s+1])\\&\le\mathbf{P}(B^\mu(r)\in\widehat{D}_1\quad\forall r\in[s,s+[M^\mu]_{s+1}]).\end{aligned} \quad (4.4)$$

By (4.3), $[M^\mu]_{s+1} \geq \delta$ a.s. for all x, s. Hence

$$\mathbf{P}(B^\mu(r) \in \widehat{D}_1 \quad \forall r \in [s, s+[M^\mu]_{s+1}]) \leq \mathbf{P}(B^\mu(r) \in \widehat{D}_1 \quad \forall r \in [s, s+\delta]). \tag{4.5}$$

By (4.1)–(4.2) and (4.4)–(4.5), it follows that

$$\sup_\mu \mathbf{P}(\tau^{x,s} > s+1) \leq \nu \triangleq \sup_\mu \mathbf{P}(B^\mu(r) \in \widehat{D}_1 \quad \forall r \in [s, s+\delta]),$$

and $\nu = \nu(n, D, c_f, \delta) \in (0,1)$. This completes the proof. $\square$

Let $\mathcal{A}^*(t)$ be the operator formally adjoint to $\mathcal{A}(t)$, i.e.,

$$\mathcal{A}^*(t)u = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k}\Big(f_k(x,t)u(x)\Big) + \frac{1}{2}\sum_{k,m=1}^{n} \frac{\partial^2}{\partial x_k \partial x_m}\Big(b_{km}(x,t)u(x)\Big). \tag{4.6}$$

Consider the boundary value problem

$$\begin{cases} \frac{\partial p}{\partial t}(x,t) = \mathcal{A}^*(t)p(x,t) + q(x,t)p(x,t) + \xi(x,t), \\ p(x,t)|_{x\in\partial D} = 0, \\ p(x,s) \equiv \rho(x). \end{cases} \tag{4.7}$$

Here $t \geq s$, $q : Q \to \mathbf{R}$ and $\rho : D \to \mathbf{R}$ are some functions, the function $q(x,t)$ is measurable and bounded, $\xi|_{D\times[s,T]} \in X_2^{-1}(s,T)$ for all $T \geq s$.

The following Proposition 4.1 presents some facts from Chapter III from Ladyzhenskaya *et al* (1968) and from Chapter III from Ladyzhenskaya (1985). Estimate (4.8) is "the energy inequality" (3.14) from Ladyzhenskaya (1985).

Proposition 4.1 *Let $0 \leq s < T$, $T - s \leq d$, where $d > 0$ is given. Assume that $(f, \beta) \in \Theta(\mathcal{P}_0)$, $\rho \in H^0$, $\xi \in X_2^{-1}(s,T)$. Then there exists the unique solution $p \in X_2^1(s,T) \cap \mathcal{C}^0(s,T)$ of the problem (4.7), and there exists a constant $C = C(\mathcal{P}_q, d)$ such that*

$$\sup_{t\in[s,T]} \|p(\cdot,t)\|^2_{H^0} + \int_s^T \|p(\cdot,t)\|^2_{H^1} dt \leq C \left(\|\rho\|^2_{H^0} + \int_s^T \|\xi(\cdot,t)\|^2_{H^{-1}} dt \right) \tag{4.8}$$

for all $(f, \beta) \in \Theta(\mathcal{P}_0)$.

In addition to Proposition 4.1, notice that $p^{(s)}(\cdot,T) = p^{(t)}(\cdot,T)$ for $s < t < T$ if $p^{(s)}(\cdot,t) = p^{(t)}(\cdot,t)$. Here $p^{(s)}$ denotes the corresponding solution of (4.7) with $\xi \equiv 0$ given s.

To proceed further, we need some auxiliary lemmas.

We assume below that conditions of Theorem 2.1 are satisfied for q, i.e., we assume that $\sup_{(x,t)\in Q} q(x,t) < -\ln\nu$, and ν is the same as in Lemma 3.1.

Lemma 4.1 *Let p be the solution of (4.7) with $\xi\equiv 0$. Then*

$$\int_D |p(x,t)|dx \le C_0\ e^{-\omega_*(t-s)}\|\rho\|_{H^0} \quad \forall t\in[s,+\infty), \tag{4.9}$$

$$|p(x,t)| \le C_1\ e^{-\omega_*(t-s)}\|\rho\|_{H^0} \quad \forall t\in[s+1,+\infty), \tag{4.10}$$

where $\omega_* \stackrel{\Delta}{=} -\ln\nu - \sup_{(x,t)\in Q} q(x,t)$, and $C_i = C_i(\mathcal{P}_q)$ are constants that do not depend on s,t,ρ and depend on $\mathcal{P}_q$ only, $i=0,1$.

Proof. By linearity of the problem, it suffices to consider ρ such that $\rho(x)\ge 0$ and $\int_D \rho(x)dx = 1$. Let $p_0(x,t) \stackrel{\Delta}{=} p(x,t)e^{-\lambda(t-s)}$ and $q_0(x,t) \stackrel{\Delta}{=} q(x,t)-\lambda$, where $\lambda \stackrel{\Delta}{=} \sup_{(x,t)\in Q} q(x,t)$. Clearly, $q_0(x,t)\le 0$ and

$$\begin{cases} \frac{\partial p_0}{\partial t}(x,t) = \mathcal{A}^*(t)p_0(x,t) + q_0(x,t)p_0(x,t), \\ p_0(x,t)|_{x\in\partial D} = 0, \\ p_0(x,s)\equiv\rho(x). \end{cases}$$

Therefore, $p_0(x,t)$ is the probability density function of the process $y^{a,s}(t)$ under assumption that this process is absorbed at ∂D and is killed inside D with the rate $|q_0(x,t)|$, where a is a random vector independent on $w(\cdot)$ with the probability density function ρ. Hence $0\le p_0(x,t)\le\pi(x,t)$, where $\pi(x,t)$ is the probability density function of the process $y^{a,s}(t)$ under assumption that this process is absorbed at ∂D *without being killed* inside D, i.e.,

$$\begin{cases} \frac{\partial \pi}{\partial t}(x,t) = \mathcal{A}^*(t)\pi(x,t), \\ \pi(x,t)|_{x\in\partial D} = 0, \\ \pi(x,s)\equiv\rho(x). \end{cases}$$

Because of absorption at ∂D, we have

$$\int_D \pi(x,t)dx \le \int_D \pi(x,r)dx \quad \forall r,t\in\mathbf{R}:\ s\le r\le t.$$

By Lemma 3.1, it follows that

$$\int_D \pi(x,t+1)dx \le \nu \int_D \pi(x,t)dx \quad \forall t \ge s.$$

Hence

$$\begin{aligned}
&\textstyle\int_D |p_0(x,t)|dx = \int_D p_0(x,t)dx \le \|\pi(\cdot,t)\|_{L_1(D)} \\
&\le \|\pi(\cdot, s+\lfloor t-s\rfloor)\|_{L_1(D)} \\
&\le \nu\|\pi(\cdot, s+\lfloor t-s\rfloor -1)\|_{L_1(D)} \\
&\le \nu^2\|\pi(\cdot, s+\lfloor t-s\rfloor -2)\|_{L_1(D)} \le \cdots \le \nu^{\lfloor t-s\rfloor}\|\rho\|_{L_1(D)} = \ e^{\lfloor t-s\rfloor \ln \nu}\|\rho\|_{L_1(D)},
\end{aligned} \tag{4.11}$$

where $\lfloor t\rfloor$ denotes the integer part of t. Then (4.9) follows.

Let us prove (4.10). Let $\Delta \triangleq \{(t,s) : \ t \ge s \ge 0\}$, and let $g(\cdot) : D^2 \times \Delta \to \mathbf{R}$ be the Green's function for the equation (4.7) such that if $\xi \equiv 0$ then

$$p(x,t) = \int_D g(x,y,t,s)p(y,s)dy, \quad t \ge s \ge 0. \tag{4.12}$$

Let $G(x,y,t,s)$ be the fundamental solution of problem (4.7) without the boundary condition on ∂D (i.e., for $D = \mathbf{R}^n$); the order of independent variables for G is similar to (4.12). By Lemma 7 from Aronson (1968), it follows that $|g(x,y,t,s)| \le |G(x,y,t,s)|$ $(\forall x,y,t,s)$. Using estimates from Aronson (1967), we obtain

$$|g(x,y,t+1,t)| \le |G(x,y,t+1,t)| \le c \quad \forall x,y \in D,\ t \ge 0, \tag{4.13}$$

where $c = c(\mathcal{P}_q)$ is a constant. By (4.11) and (4.13), it follows (4.10). This completes the proof of Lemma 4.1. $\square$

Let us introduce linear normed spaces $Z^k(s,+\infty)$ of functions $u : (s,+\infty) \to H^k$ with finite norm

$$\|u\|_{Z^k(s,+\infty)} = \sum_{m=0}^{+\infty} \Big(\int_{s+m}^{s+m+1} \|u(\cdot,t)\|^2_{H^k} dt\Big)^{1/2}.$$

Lemma 4.2 *Let $s \geq 0$, let $\rho \in H^0$, and let $\xi \in X_1^0(s,+\infty) \cup X_2^{-1}(s,+\infty)$ be such that $\xi(\cdot,t) \equiv 0$ for $t > s+1$. Then there exists the solution $p \in X_1^1(s,+\infty) \cap \mathcal{C}^0(s,+\infty)$ of problem (4.7). This solution is unique up to equivalency, and*

$$\|p\|_{X_1^1(s,s+1)} + \|p\|_{\mathcal{C}^0(s,s+1)} \leq c_1(\|\rho\|_{H^0} + \|\xi\|_{X_1^0(s,s+1)}), \tag{4.14}$$

$$\|p\|_{X_2^1(s,s+1)} + \|p\|_{\mathcal{C}^0(s,s+1)} \leq c_2(\|\rho\|_{H^0} + \|\xi\|_{X_2^{-1}(s,s+1)}), \tag{4.15}$$

$$\|p\|_{X_2^1(s+1,+\infty)} \leq c_3\|p(\cdot,s+1)\|_{H^0}, \tag{4.16}$$

$$\|p\|_{X_1^1(s+1,+\infty)} + \|p\|_{\mathcal{C}^0(s+1,+\infty)} \leq c_4\|p(\cdot,s+1)\|_{H^0}, \tag{4.17}$$

$$\|p\|_{Z^1(s+1,+\infty)} \leq c_5\|p(\cdot,s+1)\|_{H^0}, \tag{4.18}$$

where $c_i = c_i(\mathcal{P}_q) > 0$ are constants that do not depend on s and depend on $\mathcal{P}_q$ only, $i = 1,\ldots,5$.

Proof of Lemma 4.2. Let us prove (4.14). For any $T \geq s$ and any $\varepsilon \in (0,\delta)$, we have

$$\begin{aligned}
\|p(\cdot,T)\|_{H^0} - \|p(\cdot,s)\|_{H^0} &= \textstyle\int_s^T \|p(\cdot,t)\|^{-1}_{H^0}(p(\cdot,t), \mathcal{A}^*p(\cdot,t) + q(\cdot,t)p(\cdot,t) + \xi(\cdot,t))_{H^0}dt \\
&= \textstyle\int_s^T \|p(\cdot,t)\|^{-1}_{H^0}\Big\{-\frac{1}{2}\sum_{i,j=1}^n\Big[\Big(\frac{\partial p}{\partial x_i}(\cdot,t), b_{ij}(\cdot,t)\frac{\partial p}{\partial x_j}(\cdot,t)\Big)_{H^0} \\
&\qquad + \tfrac{1}{2}\Big(p(\cdot,t), \frac{\partial b_{ij}}{\partial x_j}(\cdot,t)\frac{\partial p}{\partial x_i}(\cdot,t)\Big)_{H^0}\Big] \\
&\quad + \textstyle\sum_{i=1}^n\Big(\frac{\partial p}{\partial x_i}(\cdot,t), f_i(\cdot,t)p(\cdot,t)\Big)_{H^0} + \Big(\xi(\cdot,t) + q(\cdot,t)p(\cdot,t), p(\cdot,t)\Big)_{H^0}\Big\}dt \\
&\leq \textstyle\int_s^T \|p(\cdot,t)\|^{-1}_{H^0}\sum_{i=1}^n\Big\{\frac{1}{2}(\varepsilon-\delta)\Big\|\frac{\partial p}{\partial x_i}(\cdot,t)\Big\|^2_{H^0} + c(\varepsilon)\|p(\cdot,t)\|^2_{H^0} + \|p(\cdot,t)\|_{H^0}\|\xi(\cdot,t)\|_{H^0}\Big\}dt
\end{aligned} \tag{4.19}$$

Here the constant $c(\varepsilon) = c(\varepsilon,\mathcal{P}_q) > 0$ depends only on ε and $\mathcal{P}_q \triangleq (\mathcal{P}_0, \sup q(x,t))$. We had used elementary inequality $2\alpha\beta \leq \varepsilon\alpha^2 + \varepsilon^{-1}\beta^2$ $(\forall \alpha,\beta,\varepsilon \in \mathbf{R}, \varepsilon > 0)$, and inequality

$$\Big(\frac{\partial p}{\partial x_i}(\cdot,t), F(\cdot)p(\cdot,t)\Big)_{H^0} \leq \frac{\varepsilon}{2}\Big\|\frac{\partial p}{\partial x_i}(\cdot,t)\Big\|^2_{H^0} + \frac{1}{2\varepsilon}\|F(\cdot)\|_{L_\infty(D)}\|p(\cdot,t)\|_{H^0},$$

where $F(\cdot) : D \to \mathbf{R}$ is an arbitrary measurable bounded function.

By Poincaré - Friedrichs inequality (see, e.g., Yosida (1965)), it follows that there exist a constant $\kappa = \kappa(D) > 0$ such that

$$\|p(\cdot,t)\|_{H^0}^{-1} \sum_{i=1}^{n} \left\| \frac{\partial p}{\partial x_i}(\cdot,t) \right\|_{H^0}^2 \geq \kappa \|p(\cdot,t)\|_{H^1}.$$

By (4.19), it follows that

$$\begin{aligned} \|p(\cdot,T)\|_{H^0} \quad &+ \bar{c}_1 \int_s^T \|p(\cdot,t)\|_{H^1}\, dt \\ &\leq \|\rho\|_{H^0} + \bar{c}_2 \Big(\int_s^T \|p(\cdot,t)\|_{H^0} dt + \int_s^T \|\xi(\cdot,t)\|_{H^0} dt \Big) \qquad \forall T \geq s. \end{aligned} \tag{4.20}$$

Here $\bar{c}_i = \bar{c}_i(\mathcal{P}_q)$ are constants that do not depend on $T \in [s, +\infty)$ for $i = 1, 2$. By Gronwall's inequality, inequality (4.20) applied for $T \in [s, s+1]$ implies (4.14).

Similarly (4.19)-(4.20), one can derive

$$\begin{aligned} \|p(\cdot,T)\|_{H^0}^2 + \quad &\widehat{c}_1 \int_s^T \|p(\cdot,t)\|_{H^1}^2 dt \\ &\leq \|p(\cdot,s)\|_{H^0}^2 + \widehat{c}_2 \left(\int_s^T \|p(\cdot,t)\|_{H^0}^2 dt + \int_s^T \|\xi(\cdot,t)\|_{H^{-1}}^2 dt \right) \qquad \forall T \geq s. \end{aligned} \tag{4.21}$$

Constants $\widehat{c}_i = \widehat{c}_i(\mathcal{P}_q) > 0$ do not depend on $T \in [s, +\infty)$. By Gronwall's inequality again, inequality (4.21) with $T \in [s, s+1]$ implies (4.15) (In fact, this is the estimate from Proposition 4.1, or a reformulation of "the energy inequality" (3.14) from Ladyzhenskaya (1985)).

Let us prove (4.16)-(4.18). Remind that $\xi(x,t) \equiv 0$ for $t > s+1$. By Lemma 4.1,

$$|p(x,t)| \leq C_1 \, e^{-\omega_*(t-s-1)} \|p(\cdot, s+1)\|_{H^0} \quad (\forall t \geq s+1),$$

where $C_1 = C_1(\mathcal{P}_q) > 0$ is a constant from (4.10), $\omega_* = -\ln \nu - \max q(x,t)$. Then

$$\begin{aligned} &\|p(\cdot,t)\|_{H^0} \leq C_1 e^{-\omega_*(t-s-1)} \|p(\cdot,s+1)\|_{H^0}, \\ &\int_{s+1}^{+\infty} \|p(\cdot,t)\|_{H^0}^2 dt \leq C_2 \|p(\cdot,s+1)\|_{H^0}^2, \\ &\int_{s+1}^{+\infty} \|p(\cdot,t)\|_{H^0} dt \leq C_3 \|p(\cdot,s+1)\|_{H^0}. \end{aligned} \tag{4.22}$$

Here $C_i = C_i(\mathcal{P}_q) > 0$ are constants. Then (4.16) follows from (4.21) and (4.22). Further, (4.17) follows from (4.20) and (4.22). By (4.21)-(4.22),

$$\begin{aligned}
\widehat{c}_1 \int_{s+m}^{s+m+1} \|p(\cdot,t)\|_{H^1}^2 dt &\leq \|p(\cdot, s+m)\|_{H^0}^2 + \widehat{c}_2 \int_{s+m}^{s+m+1} \|p(\cdot,t)\|_{H^0}^2 dt \\
&\leq C_1^2 \left[e^{-2\omega_*(m-1)} + \widehat{c}_2 \int_{s+m}^{s+m+1} e^{-2\omega_*(t-s-1)} dt \right] \|p(\cdot, s+1)\|_{H^0}^2 \\
&\leq C_* e^{-2\omega_* m} \|p(\cdot, s+1)\|_{H^0}^2 \qquad\qquad \forall m = 1, 2, \ldots
\end{aligned}$$

Here $C_* = C_*(\mathcal{P}_q) > 0$ is a constant that does not depend on m. Then (4.18) follows. This completes the proof of Lemma 4.2. $\square$

Note that (4.8) can be derived by the following way. Similarly (4.19)-(4.20), one can derive (4.21). By Gronwall's inequality, inequality (4.21) implies (4.8).

Let $0 \leq s < T$, let $\mathcal{Q} \triangleq D \times (s,T)$, and let $\gamma \geq 1$. Introduce linear normed spaces $\mathcal{W}_\gamma(s,T)$ of functions $u : [s,T] \to W^2_\gamma(D)$ that belong to $L_\gamma([s,T], \bar{\mathcal{B}}_1, \ell_1, W^2_\gamma(D))$ and such that $\frac{\partial u}{\partial t}$ belong to $L_\gamma([s,T], \bar{\mathcal{B}}_1, \ell_1, L_\gamma(D))$, with finite norm

$$\|u\|_{\mathcal{W}_\gamma(s,T)} = \Big(\int_s^T \|u(\cdot,t)\|^\gamma_{W^2_\gamma(D)} dt\Big)^{1/\gamma} + \Big(\int_s^T \Big\|\frac{\partial u}{\partial t}(\cdot,t)\Big\|^\gamma_{L_\gamma(D)} dt\Big)^{1/\gamma}.$$

It is easy to see that $\mathcal{W}_\gamma(s,T) \subset C([s,T]; L_\gamma(D))$, and this embedding is continuous. Moreover, $\mathcal{W}_\gamma(s,T) = W^{2,1}_\gamma(\mathcal{Q})$, meaning the natural bijection such that the norms are equivalent.

The space $W^l_\gamma(D)$ with non-integer l will be used below. It is a Banach space consisting of the elements of $W^{\lfloor l \rfloor}_\gamma(D)$ with finite norm

$$\|u\|_{W^l_\gamma(D)} \triangleq \|u\|_{W^{\lfloor l \rfloor}_\gamma(D)} + \sum_{j:|j|=\lfloor l \rfloor} \left(\int_D dx \int_D |\mathcal{D}^j_x u(x) - \mathcal{D}^j_y u(y)|^\gamma \frac{dy}{|x-y|^{n+\gamma(l-\lfloor l \rfloor)}}\right)^{1/\gamma}.$$

Here $\lfloor l \rfloor$ is the integer part of l, $j = (j_1, ..., j_n)$, where $j_k \geq 0$ are integers, $|j| = \sum_k j_k$,

$$\mathcal{D}^j_x u(x) = \frac{\partial^{|j|} u}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}.$$

(See, e.g., Ladyzhenskaya *et al* (1968), p. 70, and Adams (1975), p. 214).

Consider the boundary value problem in $\mathcal{Q}$

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + \mathcal{A}(t)u(x,t) + q(x,t)u(x,t) = -\varphi(x,t), \\ u(x,t)|_{x\in\partial D} = 0, \\ u(x,T) \equiv \Phi(x). \end{cases} \tag{4.23}$$

Here $q : \mathcal{Q} \to \mathbf{R}$, $\varphi : \mathcal{Q} \to \mathbf{R}$ and $\Phi : D \to \mathbf{R}$ are some measurable functions, the function $q(x,t)$ is bounded.

Let $\theta \in (s,T)$, and let $\mathcal{Q}_\theta \triangleq D \times (s,\theta)$.

Lemma 4.3 *Let $0 \le s < \theta < T$ and $\gamma \ge 2$. Assume that $(f,\beta) \in \Theta(\mathcal{P}_0)$, $\varphi \in X_2^{-1}(s,T)$, $\Phi \in H^0$, $T - s \le d$, and $T - \theta \ge d_0$, where $d > 0$ and $d_0 > 0$ are given. Then*

(i) There exists the unique solution $u \in \mathcal{C}^0(s,T) \cap X_2^1(s,T)$ of problem (4.23), and there exists a constant $C = C(\mathcal{P}_q, d) > 0$ such that

$$\|u\|_{\mathcal{C}^0(s,T)} + \|u\|_{X_2^1(s,T)} \le C\left(\|\Phi\|_{H^0} + \|\varphi\|_{X_2^{-1}(s,T)}\right) \tag{4.24}$$

for all $(f,\beta) \in \Theta(\mathcal{P}_0)$.

(ii) Let $\rho \in H^0$ be arbitrary, and let p be the solution of (4.7), where $\xi \equiv 0$. Then

$$(u(\cdot,T), p(\cdot,T))_{H^0} - (u(\cdot,s), p(\cdot,s))_{H^0} = -\int_s^T (\varphi(\cdot,t), p(\cdot,t))_{H^0} dt.$$

(iii) If $\varphi \in L_2(\mathcal{Q})$ and $\Phi \in H^1$, then $u \in \mathcal{C}^1(s,T) \cap X_2^2(s,T)$, and there exists a constant $C = C(\mathcal{P}_{|q|}, d) > 0$ such that

$$\|u\|_{\mathcal{C}^1(s,T)} + \|u\|_{X_2^2(s,T)} \le C\left(\|\Phi\|_{H^1} + \|\varphi\|_{L_2(\mathcal{Q})}\right) \tag{4.25}$$

for all $(f,\beta) \in \Theta(\mathcal{P}_0)$.

(iv) If $\varphi \in L_\gamma(\mathcal{Q})$ and $\Phi \in W_\gamma^{2-2/\gamma}(D) \cap H^1$, then the solution u of problem (4.23) belongs to $\mathcal{W}_\gamma(s,T)$, and there exists a constant $C = C(\mathcal{P}_{|q|}, d, \gamma) > 0$ such that

$$\|u\|_{\mathcal{W}_\gamma(s,T)} \le C\left(\|\Phi\|_{W_\gamma^{2-2/\gamma}(D)} + \|\varphi\|_{L_\gamma(\mathcal{Q})}\right) \tag{4.26}$$

for all $(f,\beta) \in \Theta(\mathcal{P}_0)$.

(v) If $\varphi \in L_\gamma(\mathcal{Q})$*, then the solution* u *is such that* $u|_{\mathcal{Q}_\theta} \in \mathcal{W}_\gamma(s,\theta)$*, and there exists a constant* $C = C(\mathcal{P}_{|q|}, d, \gamma) > 0$ *such that*

$$\|u|_{\mathcal{Q}_\theta}\|_{\mathcal{W}_\gamma(s,\theta)} \le C\left(\|\Phi\|_{H^0} + \|\varphi\|_{L_\gamma(\mathcal{Q})}\right) \tag{4.27}$$

for all $(f,\beta) \in \Theta(\mathcal{P}_0)$*.*

(vi) If $\gamma > n+2$ *and* $\varphi \in L_\gamma(\mathcal{Q})$*, then* $u(x,t)|_{\mathcal{Q}_\theta}$ *and* $\frac{\partial u}{\partial x_k}(x,t)|_{\mathcal{Q}_\theta}$*,* $k = 1,...,n$*, are continuous and belong to Hölder class* $\mathcal{H}^{1+\alpha,(1+\alpha)/2}(\mathcal{Q}_\theta)$ *for* $\alpha = 1-(n+2)/\gamma$*. Moreover, there exists a constant* $C = C(\mathcal{P}_{|q|}, d, d_0, \gamma) > 0$ *such that*

$$\langle\langle u|_{\mathcal{Q}_\theta}\rangle\rangle^{(1+\alpha)}_{\mathcal{Q}_\theta} \le C\left(\|\Phi\|_{H^0} + \|\varphi\|_{L_\gamma(\mathcal{Q})}\right)$$

for all $(f,\beta) \in \Theta(\mathcal{P}_0)$*.*

Remark 4.1 Under the assumptions of statement (iv) in Lemma 4.3, $u \in W^{2,1}_\gamma(\mathcal{Q})$, and $\|u\|_{W^{2,1}_\gamma(\mathcal{Q})} \le \text{const}\,\|u\|_{\mathcal{W}_\gamma(s,T)}$, because there is a natural bijection between $W^{2,1}_\gamma(\mathcal{Q})$ and $\mathcal{W}_\gamma(s,T)$ such that the norms are equivalent. Under assumptions of statement (v), $u|_{\mathcal{Q}_\theta} \in W^{2,1}_\gamma(\mathcal{Q}_\theta)$, and $\|u|_{\mathcal{Q}_\theta}\|_{W^{2,1}_\gamma(\mathcal{Q}_\theta)} \le \text{const}\,\|u\|_{\mathcal{W}_\gamma(s,\theta)}$.

Proof of Lemma 4.3. Statement (i) follows from inequality (3.14) from Ladyzhenskaya (1985). Statement (ii) follows from the fact that the parabolic equations in (4.7) and (4.23) are adjoint, and from the equations for $\partial u/\partial t$ and $\partial p/\partial t$. Statement (iii) follows from Theorem 1.2 from Dokuchaev (1997). (Note that statement (iii) can be also derived from Theorem 6.1 and Remark 6.3 from Ladyzhenskaya *et al* (1968)) (pp. 178-180). More precisely, this statement follows from the inequality (6.25) from Ladyzhenskaya *et al* (1968), p. 180, and from the inequality (6.29) from Ladyzhenskaya (1985). In fact, Theorem 6.1 from Ladyzhenskaya *et al* (1968) deals with a special case of (f,q), but it is not really important).

Statement (iv) is a special case of Theorem 9.1, Chapter IV, from Ladyzhenskaya *et al* (1968). Formally, this theorem requires that $\Phi \in$

$W_\gamma^{2-2/\gamma}(D)$ and $\Phi|_{\partial D} = 0$. However, these conditions can be easily replaced by our condition $\Phi \in W_\gamma^{2-2/\gamma}(D) \cap H^1$. Let us show this. Let $\Phi \in W_\gamma^{2-2/\gamma}(D) \cap H^1$. Clearly, there exists a sequence $\{\Phi_i\}_{i=1}^{+\infty} \subset C^2(D)$ such that $\Phi_i|_{\partial D} = 0$ $(\forall i)$, and $\Phi_i \to \Phi$ in both spaces $W_\gamma^{2-2/\gamma}(D)$ and H^1 as $i \to \infty$. Let u_i be the solution of problem (4.23) with $\Phi = \Phi_i$. By Theorem 9.1, Chapter IV, from Ladyzhenskaya *et al* (1968), the constant C in (4.26) does not depend on $\Phi = \Phi_i$. Therefore, the sequence $\{u_i\}_{i=1}^{+\infty}$ is a Cauchy sequence in $\mathcal{W}_\gamma(s,T)$ and has a limit in this space. By statement (iii), $u_i \to u$ in $\mathcal{C}^1(s,T)$, where $u \in \mathcal{C}^1(s,T) \cap X_2^1(s,T)$ is the solution of (4.23) given Φ. Hence $u \in \mathcal{W}_\gamma(s,T)$ and (4.26) is satisfied. This completes the proof of statement (iv).

Let us prove statement (v). Consider the following sequences:

$$
\begin{aligned}
&h_1 = 2, \qquad h_m \triangleq h_{m-1}\tfrac{n+2}{n}, \\
&\chi_1 = 2, \qquad \chi_m = 2 - \tfrac{2}{h_m}, \qquad\qquad m = 2, 3, \ldots.
\end{aligned}
$$

It is easy to see that

$$
\chi_m = 2 - \frac{n}{h_{m-1}} + \frac{n}{h_m}, \quad \chi_m > 0, \quad h_{m+1} > h_m, \qquad h_m \to \infty \quad \text{as} \quad m \to +\infty.
$$

Clearly, there exists $N = N(n)$ such that $h_N \geq \gamma$ and $h_m < \gamma$ for all $m < N$. Let $s_m \triangleq T - (m-1)(T-\theta)/N$, $m = 1, \ldots, N+1$. It is easy to see that

$$
\theta = s_{N+1} < \cdots < s_{m+1} < s_m < \cdots < s_1 = T.
$$

Let us prove that there exists a set $\{t_m\}_{m=1}^N \subset [\theta, T]$ such that

$$
\begin{aligned}
t_m \in (s_{m+1}, s_m], \quad & u(\cdot, t_m) \in W_{h_m}^2(D) \cap H^1, \\
& \|u(\cdot, t_m)\|_{W_{h_m}^2(D)} \leq C\left(\|\Phi\|_{H^0} + \|\varphi\|_{L_\gamma(Q)}\right),
\end{aligned}
\tag{4.28}
$$

where $C = C(\mathcal{P}_{|q|}, \gamma, d)$, $m = 1, \ldots, N$. Note that we allow that $\{t_m\}_{m=1}^N$ can depend on (Φ, φ).

First, let us prove that (4.28) is satisfied for $m = 1$ for some t_1. Clearly, $H^2 \subset W_2^2(D) = W_{h_1}^{\chi_1}(D)$, and this embedding is continuous.

Therefore, it suffices to prove that there exists $t_1 \in (s_2, s_1] = (s_2, T]$ such that

$$u(\cdot, t_1) \in H^2, \qquad \|u(\cdot, t_1)\|_{H^2} \le C\left(\|\Phi\|_{H^0} + \|\varphi\|_{L_2(Q)}\right), \qquad (4.29)$$

where $C = C(\mathcal{P}_{|q|}, d, d_0)$.

Let $h \triangleq (s_2 + s_1)/2 = (s_2 + T)/2$. By statement (i), it follows that $u \in \mathcal{C}^0(s, T) \cap X_2^1(s, T)$, and

$$\int_s^T \|u(\cdot, t)\|_{H^1}^2 dt \le C_1\left(\|\Phi\|_{H^0}^2 + \|\varphi\|_{L_2(Q)}^2\right),$$

where $C_1 = C_1(\mathcal{P}_q, d) > 0$. Hence

$$\inf_{r\in[h,T]} \|u(\cdot, r)\|_{H^1}^2 \le \tfrac{1}{T-h}\int_h^T \|u(\cdot, t)\|_{H^1}^2 dt \le \tfrac{C_1}{T-h}\left(\|\Phi\|_{H^0}^2 + \|\varphi\|_{L_2(Q)}^2\right).$$

By statement (iii), if $r \in [h, T]$ is such that $u(\cdot, r) \in H^1$, then $u \in \mathcal{W}_2(s, r)$, and

$$\int_s^h \|u(\cdot, t)\|_{H^2}^2 dt \le \int_s^r \|u(\cdot, t)\|_{H^2}^2 dt \le C_2\left(\|u(\cdot, r)\|_{H^1}^2 + \|\varphi\|_{L_2(Q)}^2\right),$$

where $C_2 = C_2(\mathcal{P}_{|q|}, d, d_0) > 0$. Hence

$$\begin{aligned}&\inf_{r\in[\widehat{s}_2,h]} \|u(\cdot, r)\|_{H^2}^2 \le \tfrac{1}{h-\widehat{s}_2}\int_{\widehat{s}_2}^h \|u(\cdot, t)\|_{H^2}^2 dt \\ &\le C_3\left(\inf_{r\in[h,T]} \|u(\cdot, r)\|_{H^1}^2 + \|\varphi\|_{L_2(Q)}^2\right) \le C_4\left(\|\Phi\|_{H^0}^2 + \|\varphi\|_{L_2(Q)}^2\right),\end{aligned}$$

where $\widehat{s}_2 \triangleq (s_2 + h)/2$, and where $C_i = C_i(\mathcal{P}_{|q|}, d, d_0) > 0$, $i = 3, 4$. Thus, there exists $t_1 \in (s_2, s_1]$ such that (4.29) is satisfied for $m = 1$. Hence (4.28) is satisfied for $m = 1$.

Let us show that if there exists t_k such that (4.28) is satisfied for $m = k$ with $k \in \{2, \dots, N-1\}$, then there exists t_{k+1} such that (4.28) is satisfied with $m = k + 1$.

Let us now assume that there exists $t_k \in (s_{k+1}, s_k]$ such that (4.28) holds.

By the direct embedding theorem, if $\chi \triangleq \psi - n/g + n/h > 0$ and $h > g$, then $W_g^\psi(D) \subset W_h^\chi(D)$, and the embedding is continuous (see, e.g., Theorem 7.58 from Adams (1975), p. 218; the case of bounded domain is covered by Theorem 4.26, on page 84 of the cited book; see also related comments before Theorem 7.58 and Remark 7.49 there). We have that

$$W^2_{h_{m-1}}(D) \subset W^{\chi_m}_{h_m}(D), \quad m = 2, 3, \dots \tag{4.30}$$

and the embedding is continuous. Thus, $W^2_{h_k}(D) \subset W^{\chi_{k+1}}_{h_{k+1}}(D)$, and $u(\cdot, t_k) \in W^{\chi_{k+1}}_{h_{k+1}}(D)$. Moreover, $\|u(\cdot,t)\|_{W^{\chi_{k+1}}_{h_{k+1}}(D)} \le C\|u(\cdot,t)\|_{W^2_{h_k}(D)}$ for any t such that $u(\cdot,t) \in W^2_{h_k}(D)$, where $C = C(n, D, k, \gamma) > 0$ is a constant.

Let $R_m \triangleq D \times (s, t_m)$ and $\mathcal{Q}_{s_m} \triangleq D \times (s, s_m)$, $m = 1, \dots, N+1$. By statement (iv),

$$\|u|_{R_k}\|_{\mathcal{W}_{h_k}(s,t_k)} \le C\left(\|u(\cdot,t_k)\|_{W^{\chi_k}_{h_k}(D)} + \|\varphi\|_{L_\gamma(\mathcal{Q})}\right),$$

where $C = C(\mathcal{P}_{|q|}, d, d_0, h_k) > 0$ is a constant. By this estimate and (4.30), we have

$$\begin{aligned}
\|u|_{\mathcal{Q}_{s_{k+1}}}\|_{\mathcal{W}_{h_{k+1}}(s,s_{k+1})} &\le C_1\left(\inf_{r\in[s_{k+1},t_k]}\|u(\cdot,r)\|_{W^{\chi_{k+1}}_{h_{k+1}}(D)} + \|\varphi\|_{L_\gamma(\mathcal{Q})}\right)\\
&\le C_2\left(\inf_{r\in[s_{k+1},t_k]}\|u(\cdot,r)\|_{W^2_{h_k}(D)} + \|\varphi\|_{L_\gamma(\mathcal{Q})}\right)\\
&\le C_3\left(\tfrac{1}{t_k - s_{k+1}}\int_{s_{k+1}}^{t_k}\|u(\cdot,t)\|_{W^2_{h_k}(D)}dt + \|\varphi\|_{L_\gamma(\mathcal{Q})}\right)\\
&\le C_4\left(\left[\int_{s_{k+1}}^{t_k}\|u(\cdot,t)\|^{h_k}_{W^2_{h_k}(D)}dt\right]^{1/h_k} + \|\varphi\|_{L_\gamma(\mathcal{Q})}\right)\\
&\le C_5\left(\|u\|_{\mathcal{W}_{h_k}(s,t_k)} + \|\varphi\|_{L_\gamma(\mathcal{Q})}\right)\\
&\le C_6\left(\|u(\cdot,t_k)\|_{W^{\chi_k}_{h_k}(D)} + \|\varphi\|_{L_\gamma(\mathcal{Q})}\right)\\
&\le C_7\left(\|\Phi\|_{H^0} + \|\varphi\|_{L_\gamma(\mathcal{Q})}\right),
\end{aligned} \tag{4.31}$$

where $C_i = C_i(\mathcal{P}_{|q|}, d, d_0, h_k, \gamma) > 0$ are constants, $i = 1, \dots, 7$.

Further, we have

$$\begin{aligned}
\inf_{r\in[\widehat{s}_{k+2},s_{k+1}]}\|u(\cdot,r)\|_{W^2_{h_{k+1}}(D)} &\le C_1\frac{1}{\widehat{s}_{k+1} - s_{k+2}}\left(\int_{\widehat{s}_{k+2}}^{s_{k+1}}\|u(\cdot,t)\|^{h_{k+1}}_{W^2_{h_{k+1}}(D)}dt\right)^{1/h_{k+1}}\\
&\le C_2\|u|_{\mathcal{Q}_{s_{k+1}}}\|_{\mathcal{W}_{h_{k+1}}(s,s_{k+1})},
\end{aligned} \tag{4.32}$$

where $\widehat{s}_{k+2} \triangleq (s_{k+2} + s_{k+1})/2$, and where $C_i = C_i(n, D, d, d_0) > 0$. By statement (iii),

$$u(\cdot,t) \in H^1 \quad \forall t \le t_1. \tag{4.33}$$

By (4.31)-(4.33), it follows that there exists $t_{k+1} \in (s_{k+2}, s_{k+1}]$ such that (4.28) is satisfied for $m = k+1$.

Therefore, we have proved that (4.28) is satisfied for all $m = 1, \ldots, N$.

Further, we have that $W^2_{h_1}(D) = W^{\chi_1}_{h_1}(D)$ and $W^2_{h_m}(D) \subset W^2_{h_{m-1}}(D) \subset W^{\chi_m}_{h_m}(D)$, $m = 2, 3, \ldots, N+1$, and the embedding is continuous. By statement (iv), (4.28) implies that $u|_{R_m} \in \mathcal{W}_{h_m}(s, t_m)$, and

$$\begin{aligned} \|u|_{R_m}\|_{\mathcal{W}_{h_m}(s,t_m)} &\le C_1 \left(\|u(\cdot,t_m)\|_{W^{\chi_m}_{h_m}(D)} + \|\varphi\|_{L_\gamma(\mathcal{Q})} \right) \\ &\le C_2 \left(\|u(\cdot,t_m)\|_{W^2_{h_m}(D)} + \|\varphi\|_{L_\gamma(\mathcal{Q})} \right), \quad m = 1, \ldots, N, \end{aligned}$$

where $C_i = C_i(\mathcal{P}_q, d, d_0, h_k, \gamma) > 0$ are constants, $i = 1, 2$. Remind that $\mathcal{Q}_\theta = \mathcal{Q}_{s_{N+1}} \subset R_N$ and $h_N > \gamma$. Thus, statement (v) follows from this estimate with $m = N$. This completes the proof of statement (v).

Let us prove statement (vi). Note that $u|_{\mathcal{Q}_\theta} \in \mathcal{W}_\gamma(s, \theta)$, and there is the natural bijection between $W^{2,1}_\gamma(\mathcal{Q}_\theta)$ and $\mathcal{W}_\gamma(s,\theta)$ such that the norms are equivalent. Then statement (vi) follows from (v) and from continuity of embedding of $W^{2,1}_\gamma(\mathcal{Q}_\theta)$ to the Hölder class $\mathcal{H}^{1+\alpha,(1+\alpha)/2}(\mathcal{Q}_\theta)$ with $\gamma > n+2$ (see, e.g., Lemma 3.3 of Chapter II from Ladyzhenskaya *et al* (1968)). This completes the proof of Lemma 4.3. $\square$

Proof of Theorem 2.1. Let $L^*_{s,t}: H^0 \to H^0$ be the operator such that $p(\cdot,t) = L^*_{s,t}\rho$, where p is the solution of (4.7) with $\xi \equiv 0$, and where $\rho \in H^0$, $0 \le s \le t$. By Lemma 4.2, this operator is continuous, and $\|p\|_{Z^1(s,+\infty)} \le C\|\rho\|_{H^0}$ for $p = L^*_{s,\cdot}\rho$, where $C = C(\mathcal{P}_q)$ is a a constant.

Given $\varphi \in Y^{-1}(0,+\infty)$ and $s \ge 0$, let $v(s) \in H^0$ be defined such that

$$(v(s),\rho)_{H^0} = \int_s^{+\infty} (\varphi(\cdot,t), L^*_{s,t}\rho)_{H^0} dt \quad \forall \rho \in H^0. \tag{4.34}$$

Note that $v(s) \in H^0$ is well defined for all $s \ge 0$. This can be seen from the following. Let $B_{H^0} \triangleq \{\rho \in H^0 : \|\rho\|_{H^0} \le 1\}$. By (4.18), it follows that

$$\begin{aligned}&\sup_{\rho\in B_{H^0}} \textstyle\int_s^{+\infty} (\varphi(\cdot,t), L^*_{s,t}\rho)_{H^0} dt\\ &\le \|\varphi\|_{Y^{-1}(s,+\infty)} \sup_{\rho\in B_{H^0}} \|L^*_{s,t}\rho\|_{Z^1(s,+\infty)} \le c\|\varphi\|_{Y^{-1}(0,+\infty)},\end{aligned}$$

where $c = c(\mathcal{P}_q)$ is a constant. Therefore,

$$\sup_{s\ge 0} \|v(s)\|_{H^0} \le c\|\varphi\|_{Y^{-1}(0,+\infty)}. \tag{4.35}$$

Let us show that the function $v = v(\cdot,s)$ is the unique solution of problem (2.3), and v has all desired properties.

For $s \ge 0$, set

$$B_s \triangleq \{\xi \in Y^0(s,+\infty) : \ \xi(\cdot,t) = 0 \text{ if } t \ge s+1, \quad \|\xi\|_{X_2^{-1}(s,s+1)} \le 1\}.$$

We have

$$\|v\|^2_{Y^1(0,+\infty)} = \sup_{s=0,1,2,\ldots} \sup_{\xi\in B_s} \int_s^{s+1} (v(\cdot,t),\xi(\cdot,t))_{H^0} dt.$$

Further, for $\xi \in B_s$, we have

$$\begin{aligned}&\textstyle\int_s^{s+1} (v(\cdot,t),\xi(\cdot,t))_{H^0} dt = \int_s^{+\infty} (v(\cdot,t),\xi(\cdot,t))_{H^0} dt\\ &= \textstyle\int_s^{+\infty} dt \int_t^{+\infty} (\varphi(\cdot,t), L^*_{t,r}\xi(\cdot,t))_{H^0} dr\\ &= \textstyle\int_s^{+\infty} dr \int_s^{r} (\varphi(\cdot,r), L^*_{t,r}\xi(\cdot,t))_{H^0} dt = \int_s^{+\infty} (\varphi(\cdot,r), p^{(s)}_\xi(\cdot,r))_{H^0} dr,\end{aligned}$$

where

$$p^{(s)}_\xi(\cdot,r) \triangleq \int_s^r L^*_{t,r}\xi(\cdot,t) dt$$

is the solution of (4.7) with this ξ and $\rho = 0$. By Lemma 4.2, it follows that

$$\begin{aligned}\|v\|^2_{Y^1(0,+\infty)} &= \sup_{s=0,1,2,\dots} \sup_{\xi\in B_s} \int_s^{+\infty} (\varphi(\cdot,r), p^{(s)}_\xi(\cdot,r))_{H^0} dr \\ &\le \sup_{s=0,1,2,\dots} \sup_{\xi\in B_s} \|\varphi\|_{Y^{-1}(s,+\infty)} \|p^{(s)}_\xi\|_{Z^1(s,+\infty)} \le c\|\varphi\|_{Y^{-1}(s,+\infty)},\end{aligned} \tag{4.36}$$

where $c = c(\mathcal{P}_q)$ is a constant. By this estimate and (4.35), it follows that estimate (2.4) holds for v.

By (4.35), $v \in X^0_\infty(0,+\infty)$. Let us show that $v \in \mathcal{C}^0(0,+\infty)$.

Set

$$\varphi_m(x,t) = \begin{cases} \varphi(x,t) & t \le m \\ 0 & t > m \end{cases}, \qquad m = 0,1,2,\dots.$$

Denote by $v_m(\cdot,s)$ elements of H^0 defined by (4.34) for $\varphi = \varphi_m$.

By (4.36), $v_m \in Y^1(0,+\infty)$. Further, $v_m(x,s) = 0$ for all $s \ge m$ for a.e. x. By Lemma 4.3(ii), it follows that $v_m(x,s)$ is the solution of the boundary value problem in $D \times (0,m)$

$$\begin{cases} \frac{\partial v_m}{\partial s}(x,s) + \mathcal{A}(s)v_m(x,s) + q(x,s)v_m(x,s) = -\varphi(x,s) \\ v_m(x,s)|_{x\in\partial D} = 0 \\ v_m(x,m) = 0. \end{cases} \tag{4.37}$$

Clearly, $v_m \in \mathcal{C}^0(0,+\infty)$, since $v_m|_{D\times(0,m)} \in \mathcal{C}^0(0,m)$, and $v_m(\cdot,s) = 0$ for $s \ge m$. For any $\rho \in H^0$ and $s \ge 0$, we have that

$$\begin{aligned}&(v(s) - v_m(\cdot,s),\rho)_{H^0} = \int_m^{+\infty} (\varphi(\cdot,t), L^*_{s,t}\rho)_{H^0} dt \\ &\le \|\varphi\|_{Y^{-1}(0,+\infty)} \sum_{k=m}^{+\infty} \left(\int_k^{k+1} \|L^*_{s,t}\rho\|^2_{H^1} dt\right)^{1/2} \to 0 \quad \text{as} \quad m \to +\infty,\end{aligned}$$

since

$$\sum_{k=m}^{+\infty} \left(\int_k^{k+1} \|L^*_{s,t}\rho\|^2_{H^1} dt\right)^{1/2} < +\infty.$$

Hence $v_m(\cdot,s) \to v(s)$ weakly in H^0 for all $s \ge 0$.

Let us show that $v_m(\cdot,s) \to v(\cdot,s)$ in H^0 uniformly in s from any finite interval.

Parabolic equations in (4.37) and (4.7) are adjoint. This means that

$$v_m(x,s) = \int_s^m dt \int_D g(y,x,t,s)\varphi(y,t)dy, \qquad s \le m. \tag{4.38}$$

Here $g(x, y, t, s)$ is the Green's function for problem (4.7) such that (4.12) holds. By semi-group properties of the solution of problem (4.7), we have that $g(\cdot, y, t, s) = L^*_{s+1,t}\rho_y(\cdot, s)$ for any $y \in D$ and $t > s + 1$, where $\rho_y(\cdot, s) \triangleq g(\cdot, y, s+1, s)$. Similarly to (4.13), we have that $\|\rho_y(\cdot, s)\|_{L_\infty(D)} \le c$ for all $y \in D$, $s > 0$, where $c = c(\mathcal{P}_q)$ is a constant.

Therefore, $\|\rho_y(\cdot, s)\|_{H^0} \le c_*$ for all $y \in D$, $s \in (s_1, s_2)$, where $(s_1, s_2) \subset [0, +\infty)$ is an arbitrary finite interval, $c_* = c_*(\mathcal{P}_q, s_1, s_2)$ is a constant that does not depend on $y \in D$.

Let $\varphi \in Y^0(0, +\infty)$, and let $k = 1, 2, \ldots$. By (4.38) and (4.37), we have that

$$v_{m+k}(y, s) - v_m(y, s) = \int_m^{m+k} dt \int_D \varphi(x, t) g(x, y, t, s) dx.$$

Hence

$$\begin{aligned}
\|v_{m+k}(\cdot, s) \; & -v_m(\cdot, s)\|^2_{H^0} = \int_D \Big[\int_m^{m+k} dt \int_D \varphi(x, t) g(x, y, t, s) dx\Big]^2 dy \\
& = \int_D \Big[\int_m^{m+k} (\varphi(\cdot, t), L^*_{s+1,t}\rho_y(\cdot, s))_{H^0} dt\Big]^2 dy \\
& \le \int_D \Big[\int_m^{m+k} \|\varphi(\cdot, t)\|_{H^0} \|L^*_{s+1,t}\rho_y(\cdot, s)\|_{H^0} dt\Big]^2 dy \\
& = \int_D \Big[\sum_{i=m}^{m+k} \int_i^{i+1} \|\varphi(\cdot, t)\|_{H^0} \|L^*_{s+1,t}\rho_y(\cdot, s)\|_{H^0} dt\Big]^2 dy \\
& \le \int_D \Big[\sum_{i=m}^{m+k} \Big\{\int_i^{i+1} \|\varphi(\cdot, t)\|^2_{H^0} dt\Big\}^{1/2} \Big\{\int_i^{i+1} \|L^*_{s+1,t}\rho_y(\cdot, s)\|^2_{H^0} dt\Big\}^{1/2}\Big]^2 dy \\
& \le \|\varphi\|^2_{Y^0(0,+\infty)} \int_D \Big[\sum_{i=m}^{m+k} \Big\{\int_i^{i+1} \|L^*_{s+1,t}\rho_y(\cdot, s)\|^2_{H^0} dt\Big\}^{1/2}\Big]^2 dy.
\end{aligned}$$

By (4.10),

$$\sup_{y \in D} \|L^*_{s+1,t}\rho_y(\cdot, s)\|_{H^0} \le C_1 \sup_{y \in D} \|\rho_y(\cdot, s)\|_{H^0} \, e^{-\omega_*(t-s)},$$

where $C_1 = C_1(\mathcal{P}_q) > 0$ and $\omega_* = -\ln \nu - \max q(x, t) > 0$. Hence

$$\|v_{m+k}(\cdot, s) - v_m(\cdot, s)\|_{H^0} \le \|\varphi\|_{Y^0(0,+\infty)} \sum_{i=m}^{m+k} \Big(\int_i^{i+1} C_1^2 \sup_{y \in D} \|\rho_y(\cdot, s)\|^2_{H^0} \, e^{-2\omega_*(t-s)} dt\Big)^{1/2} \to 0$$

as $m \to +\infty$ uniformly in k and $s \in [s_1, s_2]$, where $[s_1, s_2] \subset [0, +\infty)$ is an arbitrary finite interval. Hence $\{v_m|_{D \times [s_1, s_2]}\}_{m=1}^{+\infty}$ is a Cauchy sequence in $\mathcal{C}^0(s_1, s_2)$, and it converges in this space. Remind that $v_m(\cdot, s) \to v(s)$ weakly in H^0 for all $s \geq 0$. Hence $v_m|_{D\times[s_1,s_2]} \to v|_{D\times[s_1,s_2]}$ in $\mathcal{C}^0(s_1, s_2)$, and $v \in \mathcal{C}^0(0, +\infty)$ for any $\varphi \in Y^0(0, +\infty)$. The set $Y^0(0, +\infty)$ is dense in $Y^{-1}(0, +\infty)$, the space $\mathcal{C}^0(0, +\infty)$ is complete, and (4.35) holds, i.e., $\sup_{s\geq 0} \|v(\cdot, s)\|_{H^0} \leq \text{const}\, \|\varphi\|_{Y^{-1}(0,+\infty)}$. It follows that $v \in \mathcal{C}^0(0, +\infty)$ for any $\varphi \in Y^{-1}(0, +\infty)$.

Let us show that $v(x, s)$ satisfies (2.3) in the desired sense. For an arbitrary $\zeta(x) \in C^\infty(D)$, such that $\operatorname{supp}\zeta \subseteq \operatorname{int}D$, for any $\theta > t > 0$, we have

$$\begin{aligned}
&(\zeta, v(\cdot, \theta) - v(\cdot, t))_{H^0} = \lim_{m\to+\infty}(\zeta, v_m(\cdot, \theta) - v_m(\cdot, t))_{H^0} \\
&= \lim_{m\to+\infty}(\zeta, \textstyle\int_t^\theta (\mathcal{A}(r)v_m(x, r) + q(x, r)v_m(x, r) - \varphi_m(x, r))dr)_{H^0} \\
&= \lim_{m\to+\infty}\{\textstyle\int_t^\theta (A^*(r)\zeta(x) + q(x, r)\zeta(x), v_m(x, r))_{H^0}dr - (\zeta, \int_t^\theta \varphi_m(x, r)dr)_{H^0}\} \\
&= \textstyle\int_t^\theta (A^*(r)\zeta(x) + q(x, r)\zeta(x), v(x, r))_{H^0}dr - (\zeta, \int_t^\theta \varphi(x, r)dr)_{H^0}.
\end{aligned} \tag{4.39}$$

Thus, v satisfies (2.3) in the desired sense, i.e., as a generalized solution.

Let us prove uniqueness of the solution of (2.3). Let $\widetilde{v}(x, t)$ be another solution from $X^0_\infty(0, +\infty) \cap X^1_{2,loc}(0, +\infty)$. Let $\rho \in H^0$ be arbitrary, and let $p(\cdot, t) \triangleq L^*_{s,t}\rho$, where $t \geq s$. By Lemma 4.3(ii),

$$(\widetilde{v}(\cdot, T), p(\cdot, T))_{H^0} - (\widetilde{v}(\cdot, s), p(\cdot, s))_{H^0} = -\int_s^T (\varphi(\cdot, t), p(\cdot, t))_{H^0}dt \qquad \forall s, T:\ 0 \leq s \leq T.$$

Remind that $p \in Z^1(s, +\infty)$. It follows that

$$(\widetilde{v}(\cdot, s), \rho)_{H^0} = \int_s^\infty (\varphi(\cdot, t), p(\cdot, t))_{H^0}dt.$$

Since ρ was arbitrary, we have that $\widetilde{v}(\cdot, s) = v(\cdot, s)$ in H^0 (see (4.34)). This completes the proof of statements of Theorem 2.1. $\square$

Proof of Theorem 2.3. For $\varphi \in Y^0(0,+\infty)$, the solution of problem (4.37) can be presented as

$$v_m(x,s) = \mathbf{E}\xi_m(x,s), \tag{4.40}$$

where

$$\xi_m(x,s) \triangleq \int_s^{\tau_m^{x,s}} \varphi(y^{x,s}(t),t)\exp\Big\{\int_s^t q(y^{x,s}(r),r)dr\Big\}dt, \qquad \tau_m^{x,s} \triangleq \tau^{x,s}\wedge m.$$

The equality (4.40) is satisfied for all $s \geq 0$ for a.e. x. For $\varphi|_{D\times(0,m)} \in L_{n+1}(D \times (0,m))$, it follows from the generalized Itô's formula from Krylov (1985), §II.10. If $\varphi \in Y^0(0,+\infty)$, then the generalized Itô's formula from Dokuchaev (1994) can be used.

Let us prove (2.8) for $v(s) = v(\cdot,s)$ defined by (4.34). We have proved already that $v_m(\cdot,s) \to v(\cdot,s)$ in H^0 and, therefore, in $L_1(D)$, as $m \to +\infty$ for any given $s \geq 0$. By linearity of (2.3), it suffices to consider the case of $\varphi(x,t) \geq 0$. Then $\xi_m(x,s)$ is non-decreasing in m (in the sense of non-negativity in $L_1(D)$). Then (2.8) follows for $\varphi \in Y^0(0,+\infty)$ and for $v(x,s)$ defined by (4.34) for all $s \geq 0$ for a.e. x.

Proof of Theorem 2.2.

Let us prove statement (i). Let $Q_s^* \triangleq D \times (s, s+1/2)$, $Q_s \triangleq D \times (s, s+1)$. By Lemma 4.3 (v), we have that

$$\begin{aligned}\|v|_{Q_s^*}\|_{W_\gamma^{2,1}(Q_s^*)} &\le C_1 \|v|_{Q_s^*}\|_{\mathcal{W}_\gamma(s,s+1/2)} \\ &\le C\left(\|\varphi\|_{L_\gamma(Q_s)} + \|v(\cdot, s+1)\|_{H^0}\right) \le C \sup_{s\ge 0} \|\varphi\|_{L_\gamma(Q_s)}\end{aligned}$$

for all $s \ge 0$, where $C_i = C_i(\mathcal{P}_q, \gamma) > 0$ are constants. Then statement (i) follows.

Let us prove statement (iv). By Lemma 4.3 (vi), we have that

$$\langle\langle u|_{Q_s^*}\rangle\rangle_{Q_s^*}^{(1+\alpha)} \le C\left(\|\varphi\|_{L_\gamma(Q_s)} + \|v(\cdot, s+1)\|_{H^0}\right) \le C \sup_{s\ge 0} \|\varphi\|_{L_\gamma(Q_s)}$$

for all $s \ge 0$, where $C = C(\mathcal{P}_q, \gamma) > 0$ is a constant. Then statement (ii) follows. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.4. Consider first $\varphi \in L_2(Q_0)$. Instead of (2.9), consider boundary value problem (2.3) when

$$f(x,t) \equiv f(x,t+1), \quad \beta(x,t) \equiv \beta(x,t+1), \quad q(x,t) \equiv q(x,t+1), \tag{4.41}$$

and when the parabolic equation is replaced by

$$\frac{\partial v}{\partial t}(x,t) + \mathcal{A}(t)v(x,t) + \{q(x,t) + \ln|\mu|\}v(x,t) = -\widehat{\varphi}(x,t),$$

where $\widehat{\varphi}$ is such that

$$\widehat{\varphi}(x,t) \equiv \varphi(x,t)e^{-t\ln|\mu|}, \quad t \in [0,1], \qquad \widehat{\varphi}(x,t) \equiv \frac{\mu}{|\mu|}\widehat{\varphi}(x,t+1).$$

We have that

$$v(x,s) = \mathbf{E}\int_s^{\tau^{x,s}} \widehat{\varphi}(y^{x,s}(t),t)\exp\Bigl(\int_s^t \{q(y^{x,s}(r),r) + \ln|\mu|\}dr\Bigr)dt,$$

and this equality holds for all $s \geq 0$ for a.e. $x \in D$. By (4.41), the probability distribution of the vector $y^{x,s}(t)$ coincides with that of vector $y^{x,s+k}(t+k)$ for all $k = 1, 2, \ldots$. Then $\frac{\mu}{|\mu|}v(x,0) = v(x,1)$.

Set $V(x,t) \triangleq v(x,t)e^{t\ln|\mu|}$. We have that

$$\begin{aligned}
&\tfrac{\partial V}{\partial t}(x,t) + \mathcal{A}(t)V(x,t) + q(x,t)V(x,t) \\
&= \tfrac{\partial v}{\partial t}(x,t)e^{t\ln|\mu|} + \ln|\mu|V(x,t) + \mathcal{A}(t)V(x,t) + q(x,t)V(x,t) \\
&= -[\mathcal{A}(t)v(x,t) + \{q(x,t) + \ln|\mu|\}v(x,t) + \widehat{\varphi}(x,t)]e^{t\ln|\mu|} \\
&\qquad\qquad + \ln|\mu|V(x,t) + \mathcal{A}(t)V(x,t) + q(x,t)V(x,t) = \varphi(x,t).
\end{aligned}$$

Clearly, $V(x,0) \equiv v(x,0)$, and $V(x,1) \equiv |\mu|v(x,1)$. Hence $\mu V(x,0) \equiv V(x,1)$, and V is the solution of (2.9). Inequality (2.10) is satisfied with a constant c defined by the estimate for v from Theorem 2.1 (i), and this c does not depend on $\varphi \in L_2(Q_0)$.

Therefore, statements (i)-(ii) are proved for all $\varphi \in L_2(Q_0)$.

Let $\varphi \in X_2^{-1}(0,1)$. Clearly, $L_2(Q_0)$ is dense $X_2^{-1}(0,1)$, and there exists a sequence $\{\varphi_i\}_{i=1}^{+\infty} \subset L_2(Q_0)$ such that $\varphi_i \to \varphi$ in $X_2^{-1}(0,1)$ as $i \to \infty$. Let V_i be the solution of problem (2.9) with $\varphi = \varphi_i$. By statement (i) that is proved already for $\varphi_i \in L_2(Q_0)$, the sequence $\{V_i\}_{i=1}^{+\infty}$

is a Cauchy sequence in $X_2^1(0,1)$ and in $\mathcal{C}^0(0,1)$. Hence this sequence has the limit $V \in X_2^1(0,1) \cap \mathcal{C}^0(0,1)$. It is easy to see that this V is a solution of problem (2.9). Uniqueness of V follows from (2.10). Therefore, statements (i)-(ii) hold for all $\varphi \in X_2^{-1}(0,1)$.

Statement (iii) follows from Theorem 2.1(v) applied for v. This completes the proof of Theorem 2.4. $\square$

Proof of Theorem 1.1. Let e_1 and e_2 be the indicator functions of the random events $\{\tau_1 \geq \tau_2\}$ and $\{\tau_2 > \tau_1\}$ respectively.

Let $\mathcal{F}_t$ be the filtration generated by $w(t)$ and a.

The random variables e_i are measurable with respect to the σ-algebras $\mathcal{F}_{\widetilde{\tau}}$, $\mathcal{F}_{\tau_i}$, $i = 1, 2$, associated with the Markov times (with respect to the filtration $\mathcal{F}_t$) $\widetilde{\tau}$ and τ_i (see, e.g., Gihman and Skorohod (1975), Chapter 4, §2).

Set

$$\zeta_i(t) \triangleq v_i(y_i(t), t), \quad \xi_i(t) \triangleq \zeta_i(t) e^{\lambda(t-\widetilde{\tau})}, \quad t \in [\widetilde{\tau}, \tau_i].$$

Clearly, $1 \in L_\gamma(D)$ for all $\gamma > 1$. By Theorem 2.1 (iv)-(v), it follows that $v_i(x,t)$ and $\frac{\partial v_i}{\partial x_k}(x,t)$ are continuous and bounded, and the norms $\|\frac{\partial v_i}{\partial t}\|_{L_\gamma(Q_s)}$, $\|\frac{\partial^2 v_i}{\partial x_k \partial x_m}\|_{L_\gamma(Q_s)}$ are bounded in $s \geq 0$ for any $\gamma > 1$, where $Q_s \triangleq D \times (s, s+1)$, $k, m = 1, \ldots, n$. Therefore, we can apply to $\zeta_i(t)$ the generalized Itô's formula given by Theorem II.10.1 from Krylov (1980), p. 122. By this Itô's formula and (2.12), we obtain

$$\begin{aligned} d\zeta_i(t) &= \left(\left[\tfrac{\partial v_i}{\partial t}(y_i(t), t) + \mathcal{A}_i(t) v_i(y_i(t), t) \right] dt + \tfrac{\partial v_i}{\partial x}(y_i(t), t) \beta_i(y_i(t), t) dw(t) \right. \\ &= -[\lambda v_i(y_i(t), t) + 1] dt + \tfrac{\partial v_i}{\partial x}(y_i(t), t) \beta_i(y_i(t), t) dw(t) \\ &= -[\lambda \zeta_i(t) + 1] dt + \tfrac{\partial v_i}{\partial x}(y_i(t), t) \beta_i(y_i(t), t) dw(t), \end{aligned}$$

and

$$\begin{aligned} d\xi_i(t) &= e^{\lambda(t-\widetilde{\tau})} d\zeta_i(t) + \lambda e^{\lambda(t-\widetilde{\tau})} \zeta_i(t) dt \\ &= e^{\lambda[t-\widetilde{\tau}]} \tfrac{\partial v_i}{\partial x}(y_i(t), t) \beta_i(y_i(t), t) dw(t) - e^{\lambda(t-\widetilde{\tau})} dt. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{E}\{e_1\{v_1[y_1(\widetilde{\tau}), \widetilde{\tau}] - v_1[y_2(\widetilde{\tau}), \widetilde{\tau}]\}\} &= \mathbf{E}\{e_1\{v_1[y_1(\tau_2), \tau_2] - v_1[y_2(\tau_2), \tau_2]\}\} \\ &= -\mathbf{E}\{e_1\{v_1[y_1(\tau_1), \tau_1] - v_1[y_1(\tau_2), \tau_2]\}\} \\ &= -\mathbf{E}\{e_1\{\xi_1(\tau_1) - \xi_1(\widetilde{\tau})\}\} \\ &= \mathbf{E}\left\{e_1 \textstyle\int_{\widetilde{\tau}}^{\tau_1} e^{\lambda(t-\widetilde{\tau})} dt\right\} \\ &= \tfrac{1}{\lambda}\mathbf{E}\{e_1\{e^{\lambda(\tau_1-\widetilde{\tau})} - 1\}\} \\ &= \tfrac{1}{\lambda}\mathbf{E}\{e_1\{e^{\lambda(\tau_1-\tau_2)} - 1\}\}. \end{aligned} \tag{4.42}$$

Hence

$$\frac{1}{\lambda}\mathbf{E}\{e_1\{e^{\lambda(\tau_1-\tau_2)}-1\}\}\le \sup_{(x,t)\in Q}\left|\frac{dv_1}{dx}(x,t)\right|\mathbf{E}\{e_1|y_1(\widetilde{\tau})-y_2(\widetilde{\tau})|\}. \quad (4.43)$$

If we replaced the indices $1,2$ in (4.42) by 2,1, we get similarly that

$$\frac{1}{\lambda}\mathbf{E}\{e_2\{e^{\lambda(\tau_2-\tau_1)}-1\}\}\le \sup_{(x,t)\in Q}\left|\frac{dv_2}{dx}(x,t)\right|\mathbf{E}\{e_2|y_1(\widetilde{\tau})-y_2(\widetilde{\tau})|\}. \quad (4.44)$$

Clearly,

$$\mathbf{E}[e^{\lambda|\tau_1-\tau_2|}-1]=\mathbf{E}\{e_1\{e^{\lambda(\tau_1-\tau_2)}-1\}\}+\mathbf{E}\{e_2\{e^{\lambda(\tau_2-\tau_1)}-1\}\}. \quad (4.45)$$

Now the desired estimate follows from (4.42)-(4.45). $\square$

Remark 4.2 In fact, the condition in (1.2) that $\partial f/\partial x$ is locally bounded can be lifted. Without this condition, equation (2.7) has an unique weak solution for any given (s,a). More precisely, there exists a set $\Big(\Omega,\mathcal{F},\mathbf{P},w(\cdot),y^{a,s}(\cdot)\Big)$ such that equation (2.7) holds and $w(\cdot)$ does not depend on a; the distribution of $y^{a,s}(\cdot)$ is uniquely defined (see, e.g., Chapter II from Krylov (1980), Section 3 of Chapter 3 from Gihman and Skorohod (1975), and Theorems 4.1 and 4.3-4.4 from Dokuchaev (1997)). In this case, the formulations of the results need to be adjusted as the following. Lemma 3.1 holds for any $y^{a,s}(t)$ such as described here. Theorem 2.1 (iii) holds for $y^{x,s}(t)$ defined in the conditional probability space as $y^{a,s}(t)$ given $a=x$, where a is such that it has the probability density function in H^0. Theorem 2.1 (i)-(ii), (iv)-(v) and Theorem 1.1 hold in their present form. Remind that Theorem 1.1 requires that (1.1) is satisfied for $y_i(t)$ with the same $w(\cdot)$ for $i=1,2$.

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Further research plans in stochastic analysis: SPDEs

(i) I study backward SPDEs.

(ii) I study stochastic partial differential equations of parabolic type. First, I obtained an estimate being an analog of "the second energy inequality", or "the second fundamental inequality". If the domain is bounded, then this result is new even for the case of smooth coefficients of the parabolic Itô's equation.

(iii) I study also the difficult case of discontinuous coefficients for SPDEs. Solvability, uniqueness, and a prior estimates similar to the second fundamental inequality are obtained for bounded and unbounded domains using the technique of backward stochastic partial differential equations. For the case of discontinuous coefficients, some Cordes type conditions that ensure solvability are suggested.