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BOUNDARY VALUE PROBLEMS FOR FUNCTIONALS OF ITÔ PROCESSES*

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1. Formulation of the problem and main assumptions. Let us consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\Omega = \{\omega\}$ is a set of elementary events, $\mathcal{F}$ is some $\mathbf{P}$-complete σ-algebra of events, $\mathbf{P}$ is a probability measure on $\mathcal{F}$. We consider a standard d_0-dimensional Wiener process $W(t) = \|w_1(t), \cdots, w_{d_0}(t)\|$ with independent components. The part of this process $\|w_1(t), \cdots, w_d(t)\|$, where $d \leqq d_0$, is denoted by $w(t)$. The process $w(t)$ generates the filtration of $\mathbf{P}$-complete σ-algebras $\mathcal{F}_t = \sigma[w(s), s \leqq t] \subset \mathcal{F}$ in the usual way.

We consider an n-vector Itô stochastic differential equation

$$dy^{x,s}(t,\omega) = f\left[y^{x,s}(t,\omega), t, \omega\right] dt + \beta\left[y^{x,s}(t,\omega), t, \omega\right] dW(t), \tag{1.1}$$

$$y^{x,s}(s,\omega) = x, \tag{1.2}$$

where $0 \leqq s \leqq t \leqq T$, $x \in \mathbf{R}^n$, and the number $T > 0$. The functions $f(x,t,\omega)$: $\mathbf{R}^n \times \mathbf{R}^+ \times \Omega \to R^n$, $\beta(x,t,\omega)$: $\mathbf{R}^n \times \mathbf{R}^+ \times \Omega \to \mathbf{R}^{n\times d}$ are progressively measurable with respect to the filtration of σ-algebras $\mathcal{F}_t$ for any $x \in \mathbf{R}^n$. These functions are measurable, bounded, satisfy the global Lipschitz condition in x uniformly in t, ω, and are continuous in x, t for any ω. By a solution of (1.1), (1.2) we shall mean a "strong" solution.

Let a region $D \subset \mathbf{R}^n$ be given, and let either $D = \mathbf{R}^n$ or the region D be simply connected, bounded, and have a C^2-smooth boundary. Let us consider the cylinder $Q = D \times (0,T)$, and, for each $(x,s) \in \overline{Q}$, the random variable $\tau^{x,s}(\omega) = T \wedge \inf\{t\colon\, y^{x,s}(t,\omega) \notin \overline{D}\}$, that is, the first exit time from the set $\overline{Q} = Q \cup \partial Q$ for the vector $[y^{x,s}(t,\omega), t]$. If $D = \mathbf{R}^n$, then $\tau^{x,s}(\omega) \equiv T$.

This paper is devoted to the study of functionals of the form

$$v(x,s,\omega) = \mathbf{E}\left\{ \int_s^{\tau^{x,s}(\omega)} \varphi\left[y^{x,s}(t,\omega), t, \omega\right] dt \mid \mathcal{F}_s \right\}. \tag{1.3}$$

Here the functions $\varphi(x,t,\omega)$: $\mathbf{R}^n \times \mathbf{R}^+ \times \Omega \to \mathbf{R}$ are progressively measurable with respect to the filtration $\mathcal{F}_t$ for any $x \in \mathbf{R}^n$; $\mathbf{E}\{\cdot \mid \mathcal{F}_s\}$ is the conditional expectation.

For distributions of such functionals of Itô processes, which are not Markov, estimates are given in [1, Chap. II].

The goal of the paper consists in the representation of functionals (1.3) by solutions of special boundary value problems for stochastic partial differential equations introduced in §2. In §3 we establish the duality of these problems to boundary value problems for Itô parabolic equations which allows us to obtain supplementary information about solutions of boundary value problems of both forms (Theorem 3.2 and Theorem 4.1). Sufficient conditions for a representation of a solution of the boundary

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value problem in the form (1.3) are obtained in §2, sufficient conditions for a representation of the functional (1.3) in the form of a solution of a boundary value problem are obtained in §5 (these cases are different because the function φ does not coincide with the free term of the partial equation if the process $y^{x,s}(t,\omega)$ is not Markov). A certain smoothness of the functionals (1.3) in x, s is also established (Theorem 5.1).

Let us make additional assumptions.

For $j = 1, \cdots, d_0$ we denote by β_j the corresponding columns of the matrix β. In the case $d < d_0$ we denote by $\tilde{\beta}$ the $n \times (d_0 - d)$-matrix $\|\beta_{d+1}, \cdots, \beta_{d_0}\|$. We assume that the eigenvalues of the matrices $\beta\beta^T$ and $\tilde{\beta}\tilde{\beta}^T$ (in the case $d < d_0$) are separated from zero uniformly in all arguments.

Let us fix an integer number $r \geqq 0$ and a number $l > 0$ such that $r < l < r+1$, $r = [l]$. Let, as in [3, p. 7], $H^{l,l/2}(\overline{Q})$ be the same Banach space of functions on $\overline{Q}$ which are Hölder continuous together with r derivatives in x and $[r/2]$ derivatives in t. We assume that the functions $f(x,t,\omega)$ and $\beta(x,t,\omega)$ belong componentwise to $H^{l,l/2}(\overline{Q})$ for every $\omega \subset \Omega$ and their norms in this space are bounded uniformly in $\omega \in \Omega$. For $r < 2$ the partial derivatives of the components of the matrix $\beta(x,t,\omega)$ of second order in x are assumed to be uniformly bounded in x, t, ω. In the case $r > 0$ and $D \neq \mathbf{R}^n$ we assume that the boundary ∂D belongs to the class H^{l+2} (see [3, p. 9]). For $D = \mathbf{R}^n$ we have $\partial D = \varnothing$ and by $\overline{D}$ and $\overline{Q}$ we mean $\mathbf{R}^n$ and $\mathbf{R}^n \times [0,T]$, respectively.

Below, $L_2(D)$, $L_2(Q)$, $W_2^m(D)$, $\overset{\circ}{W}{}_2^1(\overline{D})$, $C^m(\overline{D})$, $C(\overline{Q})$, and so on denote the usual spaces ([2]–[5]) of real-valued functions on $\overline{D}$ or $\overline{Q}$. For a Banach space $\mathcal{X}$ the symbol $\|\cdot\|_{\mathcal{X}}$ denotes the norm, for a Hilbert space $\mathcal{X}$ the symbol $(\cdot,\cdot)_{\mathcal{X}}$ denotes the scalar product. For a region $G \subset \mathbf{R}^m$ the symbol $C(\overline{G} \to \mathcal{X})$ denotes the Banach space of continuous bounded functions $u\colon \overline{G} \to \mathcal{X}$ with the usual norm. $C^{m,q}(\overline{Q} \to \mathcal{X})$ denotes the set of fumctions $u(x,t)\colon \overline{Q} \to \mathcal{X}$ belonging to $C(\overline{Q} \to \mathcal{X})$ together with the first derivatives in x and q derivatives in t.

Let us consider the positive self-dual unbounded operator $\Lambda\colon L_2(D) \to L_2(D)$ of the form $\Lambda = \sqrt{I - \Delta}$, where I is the identity operator and Δ is the n-dimensional Laplace operator. For $k = 0, \pm 1$ we introduce the Hilbert spaces H^k with the scalar product $(u,v)_{H^k} = (\Lambda^k u, \Lambda^k v)_{L_2(D)}$. We assume that H^{-1} is the completion of $L_2(D)$ in the norm $\|\cdot\|_{H^{-1}}$, $H^0 = L_2(D)$, $H^1 = W_2^1(\mathbf{R}^n)$ for $D = \mathbf{R}^n$, and $H^1 = \overset{\circ}{W}{}_2^1(D)$ for $D \neq \mathbf{R}^n$. The coincidence of the corresponding norms for $k = 0, 1$ can be easily verified (see the description of H^k in [2]). For $u \in H^1$ and $v \in H^{-1}$ by $(u,v)_{H^0}$ we mean $(\Lambda u, \Lambda^{-1} v)_{H^0}$.

The symbol λ_1 denotes the Lebesgue measure in $[0,T]$. $\overline{\mathcal{P}}$ (and $\overline{\mathcal{P}}_s$ for a given $s \in [0,T]$) denotes the completion in the measure $\lambda_1 \in \mathcal{P}$ of the σ-algebra of subsets of the set $[0,T] \times \Omega$ generated by stochastic processes which are progressively measurable with respect to the filtration $\mathcal{F}_t$ (respectively, of the σ-algebra generated by measurable processes $\xi(t,\omega)$ for all $t \in [0,T]$ which are measurable with respect to $\mathcal{F}_s$).

For integer numbers $m \geqq 0$, $k = 0, \pm 1$, we introduce the Hilbert spaces

$$
\begin{aligned}
\mathfrak{L}_2 &= L^2\big([0,T] \times \Omega,\ \overline{\mathcal{P}},\ \lambda_1 \times \mathbf{P},\ \mathbf{R}\big),\\
X^k &= L^2\big([0,T] \times \Omega,\ \overline{\mathcal{P}},\ \lambda_1 \times \mathbf{P},\ H^k\big),\\
\overline{X}^k &= L^2\big([0,T] \times \Omega,\ \overline{\mathcal{P}}_T,\ \lambda_1 \times \mathbf{P},\ H^k\big),\\
\mathcal{W}^m &= L^2\big([0,T] \times \Omega,\ \overline{\mathcal{P}},\ \lambda_1 \times \mathbf{P},\ W_2^m(D)\big),\\
\overline{\mathcal{W}}^m &= L^2\big([0,T] \times \Omega,\ \overline{\mathcal{P}}_T,\ \lambda_1 \times \mathbf{P},\ W_2^m(D)\big).
\end{aligned}
$$

For $p \geqq 1$, $s \in [0,T]$ and the number l fixed above, we introduce the Banach spaces

$$\begin{aligned}
\overline{\mathcal{H}}^l &= L^2(\Omega,\ \mathcal{F}_T,\ \mathbf{P},\ H^{l,l/2}(\overline{Q})),\\
\mathcal{C}_p^m &= L^p([0,T]\times\Omega,\ \overline{\mathcal{P}},\ \lambda_1\times P,\ C^m(\overline{D})),\\
\overline{\mathcal{C}}_p^m(s) &= L^p([0,T]\times\Omega,\ \overline{\mathcal{P}}_s,\ \lambda_1\times P,\ C^m(\overline{D})),\\
C_0 &= C([0,T]\to L^2(\Omega,\ \mathcal{F}_T,\ \mathbf{P},\ L_2(D))),\\
\overline{\mathfrak{C}} &= L^2(\Omega,\ \mathcal{F}_T,\ \mathbf{P},\ C(\overline{Q})).
\end{aligned}$$

For integer numbers $m \geqq 0$, $q \geqq 0$, the symbol $\overline{\mathcal{C}}^{m,q}$ denotes the set of functions $u(x,t,\omega)$ belonging to $\overline{\mathcal{C}}$ together with the first m derivatives in x, and q derivatives in t (the derivatives must exist with probability 1).

We assume that $C_0 \subset X^0 \subset X^{-1}$, $X^1 \subset \mathcal{W}^1 \subset X^0 = \mathcal{W}^0$, $\overline{\mathcal{H}}^l \subset \overline{\mathcal{C}} \subset \overline{\mathcal{C}}_2^r(T)$, and so on, meaning the natural dense embedding. Moreover, $\mathcal{C}_p^m \subset \overline{\mathcal{C}}_p^m(T)$, $X^k \subset \overline{X}^k$, and so on. $\mathcal{H}^l$ denotes the set $\overline{\mathcal{H}}^l \cap \mathcal{W}^r$, where $r = [l]$.

We introduce the set $\partial_0 Q \subset \partial Q$ and the set $\partial_T Q \subset \partial Q$ of the following form:

$$\partial_0 Q = \{\partial D\times[0,T]\}\cup\{D\times\{0\}\},\qquad \partial_T Q = \{\partial D\times[0,T]\}\cup\{D\times\{T\}\};$$

in the case $D \neq \mathbf{R}^n$, $\partial_0 Q = \mathbf{R}^n \times \{0\}$, in the case $D = \mathbf{R}^n$, $\partial_T Q = \mathbf{R}^n \times \{T\}$.

For every $\omega \in \Omega$ we define the differential operator

$$A = A(x,t,\omega) = \sum_{i=1}^n f_i(x,t,\omega)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^n b_{ij}(x,t,\omega)\frac{\partial^2}{\partial x_i \partial x_j}. \tag{1.4}$$

Here f_i, x_i, b_{ij} are components of the vectors f, x, and of the matrix $b = \beta\beta^T$. $A^*(x,t,\omega)$ will denote the differential operator dual to the operator (1.4) (in the Lagrange sense (see [4, p. 141])).

For $g \in \mathcal{H}^l$ we consider the following boundary value problem in Q:

$$\frac{\partial U}{\partial t}(x,t,\omega) + A(x,t,\omega)U(x,t,\omega) = -g(x,t,\omega), \tag{1.5}$$

$$U(x,t,\omega)\big|_{(x,t)\in\partial_T Q} = 0. \tag{1.6}$$

We introduce the operator $\overline{\mathcal{T}}$, which maps the function g to a solution $U = \overline{\mathcal{T}}g$ of the boundary value problem (1.5)–(1.6). From [3] (see also [2] and [4]) it follows that the operators $\overline{\mathcal{T}}$: $X^0 \to \mathcal{W}^2$, $\overline{\mathcal{T}}$: $X^{-1} \to \overline{X}^1$, $\overline{\mathcal{T}}$: $X^{-1} \to C_0$ are continuous. Moreover, $U = \overline{\mathcal{T}}g \in \mathfrak{C}^{r+2,1}$ if $g \in \mathcal{H}^l$.

2. Representation of solutions of boundary value problems in the form of functionals of Itô processes. In the cylinder Q we consider the following boundary value problem for a stochastic partial differential equation:

$$d_t v(x,t,\omega) + \left[A(x,t,\omega)v(x,t,\omega) + g(x,t,\omega)\right]dt = \chi(x,t,\omega)\,dw(t), \tag{2.1}$$

$$v(x,t,\omega)\big|_{(x,t)\in\partial_T Q} = 0. \tag{2.2}$$

Here the function v is scalar-valued and values of the function χ are row d-vectors, $\chi = \|\chi_1,\cdots,\chi_d\|$. Equation (2.1) in combination with a boundary condition at $t = T$

means, in the case $v \in \mathcal{C}_2^2 \cap C_0$, $g \in X^0$, $\chi_j \in X^0$, that for any t for a.e. (almost every) x, ω,

$$(2.3)\qquad v(x,t,\omega) = \int_t^T \big[A(x,\rho,\omega)v(x,\rho,\omega) + g(x,\rho,\omega)\big]d\rho - \int_t^T \chi(x,\rho,\omega)\,dw(\rho).$$

The stochastic integral with respect to $d\omega_j(\rho)$ of a square-summable progressively measurable with respect to the filtration $\mathcal{F}_\rho$ random function is meant to be the Itô integral. This integral is believed to be extended in the standard way to an isometric operator mapping $\mathfrak{L}_2 = L^2([0,T]\times\Omega,\ \overline{\mathcal{P}},\ \lambda_1\times\mathbf{P},\ \mathbf{R})$ into $L^2(\Omega,\ \mathcal{F}_T,\ \mathbf{P},\ \mathbf{R})$. For an arbitrary function (equivalence class) in $\mathfrak{L}_2$ the value of the integral is, by definition, an equivalence class in $L^2(\Omega,\ \mathcal{F}_T,\ \mathbf{P},\ \mathbf{R})$ containing the integral of a progressively measurable representative which always exists [2, p. 11] in a class of $\mathfrak{L}_2$. The stochastic integral in (2.3) is defined for every t for a.e. x as an element of $L^2(\Omega,\ \mathcal{F}_T,\ \mathbf{P},\ \mathbf{R})$.

THEOREM 2.1. *For any function $g \in \mathcal{H}^l$ a pair of functions v, χ, where $v \in X^1 \cap C_0 \cap \mathcal{C}_2^{r+2}$, $r = [l] < l$, $\chi = \|\chi_1,\cdots,\chi_d\|$, $\chi_j \in X^0$, $j = 1,\cdots,d$, is defined satisfying* (2.1)–(2.2). *Moreover, relation* (2.2) *holds for $t = T$ for a.e. $(x,\omega) \in D\times\Omega$, and for $D \neq \mathbf{R}^n$ and $x \in \partial D$ for a.e. $(t,\omega) \in [0,T]\times\Omega$. These functions v, χ_j are determined uniquely up to equivalence (as elements of X^0).*

Let us note that the Bismut backward equations [5], which occur in the control theory for ordinary Itô equations, have a form analogous to (2.1)–(2.2): one must find a solution of an Itô equation adapted to a nondecreasing (unlike the backward equations of [2, p. 36]) filtration of σ-algebras which takes on a given (for example, nonrandom) value at a finite time. Usually this problem is solvable for the only possible diffusion coefficient which must be found in the course of the solution (thus under the conditions of Theorem 2.1 in view of uniqueness of χ for nonrandom f, β, g, we have $\chi \equiv 0$). Itô equations in an infinite-dimensional phase space, in particular parabolic Itô equations, are by now well investigated (see, for example, [2] and [6]–[18] and their bibliographies). The corresponding infinite-dimensional analogues of the Bismut equations have practically not been studied at though they were introduced in [19].

We introduce the operators $\mathcal{T}$, $\mathcal{G}$, $\mathcal{G}_j$, $j = 1,\cdots,d$, which map a function g into the functions $v = \mathcal{T}g$, $\chi = \mathcal{G}g$, $\chi_j = \mathcal{G}_j g$, respectively, satisfying (2.1)–(2.2).

THEOREM 2.2. *The operator $\mathcal{T}$ can be extended from the set $\mathcal{H}^l$ which is everywhere dense in X^0 and in X^{-1} to continuous linear operators $\mathcal{T}$: $X^{-1} \to X^1$, $\mathcal{T}$: $X^{-1} \to C_0$, $\mathcal{T}$: $X^0 \to \mathcal{W}^2$.*

In what follows the continuity of some operator signifies the possibility of its continuous extension from some everywhere dense set. The operators $\mathcal{T}$, $\mathcal{G}_j$ and others are the corresponding continuous extensions to X^{-1} (or in stipulated cases to X^0 or X^{-1}). An assertion of the type "$v = \mathcal{T}g \in \mathcal{C}_2^0$ for $g \in \mathcal{H}^l$ and operator $\mathcal{T}$: $X^{-1} \to X^1$" means that v and g are representatives with the required properties for the functions (classes) $v = \mathcal{T}g \in X^1$, $g \in X^{-1}$.

THEOREM 2.3. *The operators $\mathcal{G}$: $X^{-1} \to X^0$, $j = 1,\cdots,d$, are continuous.*

DEFINITION. A generalized solution of the problem (2.1)–(2.2) for $g \in X^{-1}$ is a pair of functions v, χ, where $v = \mathcal{T}g \in X^1 \cap C_0$, $\chi = \mathcal{G}g = \|\chi_1,\cdots,\chi_d\|$, $\chi_j \in X^0$.

THEOREM 2.4. *The operators $\mathcal{G}_j$: $X^0 \to \mathcal{W}^1$, $j = 1,\cdots,d$, are continuous. Moreover, $\mathcal{G}_j g \in X^1$ for $g \in X^0$ (we recall that $X^1 = \mathcal{W}^1$ for $D = \mathbf{R}^n$ and $X^1 \subset \mathcal{W}^1$).*

THEOREM 2.5. *Assume $f \in \mathcal{C}_2^2$, $\beta \in \mathcal{C}_2^2$, let the function $g \in \mathcal{C}_2^2$ be a representative of some class in X^{-1}, and for the equivalence classes $\mathcal{T}g \in X^0$, $\mathcal{G}_j g \in X^0$, let there* *exist representatives* $v \in \mathcal{C}_2^4 \cap C_0$, $\chi_j \in \mathcal{C}_2^2$. *Then the function*

$$\varphi(x,t,\omega) = g(x,t,\omega) - \sum_{j=1}^{d} \beta_j(x,t,\omega) \frac{\partial \chi_j}{\partial x}(x,t,\omega)$$

belongs to $\mathcal{C}_0^2$, *and, for* v *and* φ, *relation* (1.3) *holds for any* $s \in [0,T]$ *for a.e.* $(x,\omega) \in [0,T] \times \Omega$ *and for a.e.* $(s,\omega) \in [0,T] \times \Omega$ *for any* $x \in D$.

We introduce the operator B: $X^0 \to X^0$ by the formula

$$Bg = -\sum_{j=1}^{d} \beta_j(x,t,\omega) \frac{\partial \chi_j}{\partial x}(x,t,\omega), \qquad \text{where} \quad \chi_j = \mathcal{G}_j g. \tag{2.4}$$

THEOREM 2.6. *Let* $r > (n/2) + 2$, $g \in \mathcal{H}^l$ (*the numbers* $l, r = [l]$ *are fixed in Section* 1). *Then the hypotheses of Theorem* 2.5 *hold and there exist representatives* $v \in \mathcal{C}_2^4 \cap \mathcal{C}_0^0$, $\varphi \in \mathcal{C}_2^0$ *of the equivalence classes* $\mathcal{T}g \in X^0$, $g + Bg \in X^0$ *for which relation* (1.3) *holds for any* s *for a.e.* x, ω *and for a.e.* s, ω *for any* x.

Thus the solution $\mathcal{T}g$ of the boundary value problem is represented in the form (1.3), where

$$\varphi = g + Bg. \tag{2.5}$$

The question arises whether (2.5) is solvable with respect to g for a given φ.

Proof of Theorem 2.1. For a solution of the problem (1.5)–(1.6) we have $U \in \overline{\mathfrak{C}}^{r+2,1} \subset C([0,T] \to L^p(\Omega,\ \mathcal{F}_T,\ \mathbf{P},\ C^{r+2}(\overline{D}))$, $p = 1, 2$. For $\xi \in L^1(\Omega,\ \mathcal{F}_T,\ \mathbf{P},\ C^{r+2}(\overline{D}))$ the symbol $\mathcal{E}_{\mathcal{F}_s}\xi$ denotes the projection of ξ [20] to the space $L^1(\Omega,\ \mathcal{F}_s,\ \mathbf{P},\ C^{r+2}(\overline{D}))$. We introduce the functions $v(x,t,\omega) = \mathcal{E}_{\mathcal{F}_\tau} U(x,t,\omega)$ and $u(x,t,s,\omega) = \mathcal{E}_{\mathcal{F}_s} U(x,t,\omega)$. We have $v \in \mathcal{C}_1^{r+2}$, $u(\cdot,s,\cdot) \in \overline{\mathcal{C}}_1^{r+2}(s)$.

Below let the symbol $\mathcal{D}_x^l$ denote any partial derivative in x of order l, $0 \leqq l \leqq r+2$, and let the symbol $\mathcal{D}$ denote either $\mathcal{D}_x^l$ or $\partial/\partial t$.

By the Clark theorem (see [21, p. 178]) we have the representation

$$\mathcal{D}U(x,t,\omega) = \mathbf{E}\mathcal{D}U(x,t,\omega) + \sum_{j=1}^{d} \int_0^T \gamma_j^{\mathcal{D}}(x,t,\rho,\omega)\, dw_j(\rho) \qquad \text{a.s.} \tag{2.6}$$

Here $\gamma_j^{\mathcal{D}}$ are some functions of the class $C(\overline{Q} \to \mathfrak{L}_2)$ (since $\mathcal{D}U \in C(\overline{Q} \to L^2(\Omega,\ \mathcal{F}_T,\ \mathbf{P},\ \mathbf{R})))$; the order of arguments is such that $\overline{Q} = \{(x,t)\}$.

Let γ_i denote the functions in (2.6) for $\mathcal{D}U = U$ (that is, $\gamma_i = \gamma_i^{\mathcal{D}_x^0}$). It can be easily seen that all other $\gamma_j^{\mathcal{D}}$ are the derivatives of the form $\mathcal{D}\gamma$ of the functions γ_j: $\overline{Q} \to \mathfrak{L}_2$, and $\gamma_j \in C^{2,1}(\overline{Q} \to \mathfrak{L}_2)$. Below the partial derivatives $\mathcal{D}\gamma_j$ which occur, for example, in the expression $A(x,t,\omega)\gamma_j(x,t,\rho,\omega)$ are assumed to be the functions $\gamma_j^{\mathcal{D}}$.

Let us prove that the function v introduced above and the functions

$$\begin{gathered} \chi_j(x,t,\omega) = \gamma_j(x,0,t,\omega) - \int_0^t A(x,\rho,\omega)\gamma_j(x,\rho,t,\omega)\, d\rho, \\ \chi = \|\chi_1, \cdots, \chi_d\|, \end{gathered} \tag{2.7}$$

are the ones required.

We have

$$\mathcal{D}_x^l v(x,t,\omega) = \mathbf{E}\{\mathcal{D}_x^l U(x,t,\omega) \mid \mathcal{F}_t\}, \qquad \mathcal{D}u(x,t,s,\omega) = \mathbf{E}\{\mathcal{D}U(x,t,\omega) \mid \mathcal{F}_s\} \quad \text{a.s.},$$

$U \in X^1 \cap \mathcal{C}_2^{r+2}(T)$. Thus the functions v and $\mathcal{D}_x^1 v$ are square-summable in x, t, ω and the function $\|v\|_{C^{r+2}(\overline{D})}$ in t, ω, that is, $v \in X^1 \cap \mathcal{C}_2^{r+2}$. Obviously $v(x,t,\omega) - v(x,s,\omega) = \zeta_1 + \zeta_2$, where $\zeta_1 = u(x,t,t,\omega) - u(x,t,s,\omega)$ and $\zeta_2 = u(x,t,s,\omega) - u(x,s,s,\omega)$. Letting $t - s$ tend to $0+$, we have

$$\mathbf{E}\|\zeta_1\|^2_{L_2(D)} \leqq \mathbf{E}\big\|U(x,t,\omega) - U(x,s,\omega)\big\|^2_{L_2(D)} \longrightarrow 0,$$

$$\mathbf{E}\|\zeta_2\|^2_{L_2(D)} \leqq \sum_{j=1}^{d} \mathbf{E}\left\| \int_s^t \gamma_j(x,s,\rho,\omega)^2 \, d\rho \right\|_{L_1(D)} \longrightarrow 0.$$

Consequently, $v \in C_0$. By (1.6), relation (2.2) holds for a.e. x, ω for $t = T$ and for a.e. t, ω for $x \in \partial D$, $D \neq \mathbf{R}^n$. In (2.7) the coefficients of one derivatives in $A(x,\rho,\omega)$ are bounded, continuous, and $\mathcal{F}_\rho$-adapted for a.e. ω; hence $\chi_j \in X^0$.

By virtue of (1.5)–(1.6) and (2.6) we have, for a.e. x, ω,

$$\begin{aligned}
v(x,t,\omega) &= \mathbf{E}\{U(x,t,\omega) \mid \mathcal{F}_t\} \\
&= u(x,0,t,\omega) - \int_0^t \big[A(x,s,\omega)\, u(x,s,t,\omega) + g(x,s,\omega)\big]\, ds \\
&= v(x,0,\omega) + \sum_{j=1}^{d} \int_0^t \gamma_j(x,0,\rho,\omega)\, dw_j(\rho) \\
&\quad - \int_0^t \left\{ A(x,s,\omega) \left[v(x,s,\omega) + \sum_{j=1}^{d} \int_s^t \gamma_j(x,s,\rho,\omega)\, dw_j(\rho) \right] + g(x,s,\omega) \right\} ds \\
&= v(x,0,\omega) - \int_0^t \big[A(x,s,\omega) v(x,s,\omega) + g(x,s,\omega)\big]\, ds \\
&\quad + \sum_{j=1}^{d} \left\{ \int_0^t \gamma_j(x,0,\rho,\omega)\, dw_i(\rho) - \int_0^t ds \int_s^t A(x,s,\omega) \gamma_j(x,s,\rho,\omega)\, dw_j(\rho) \right\}.
\end{aligned}$$

The sum of Itô integrals in the right-hand side of the latter equality is equal to $\int_0^t \chi(\rho)\, d\omega(\rho)$ by (2.7) and the Fubini theorem for stochastic integrals (see [22]). So, for v, χ, relation (2.3) holds and v, χ are the ones required.

We introduce the operator $\mathcal{T}^*$: $X^0 \to X^0$ by the rule $\mathcal{T}^* h = \pi$, where the function $\pi \in X^1 \cap C_0$ is a solution of the boundary value problem

$$\frac{\partial \pi}{\partial t}(x,t,\omega) = A^*(x,t,\omega)\pi(x,t,\omega) + h(x,t,\omega), \qquad \pi(x,t,\omega)\big|_{(x,t)\in\partial_0 Q} = 0. \tag{2.8}$$

The operator $\mathcal{T}^*$: $X^0 \to X^0$ (and even the operator $\mathcal{T}^*$: $X^{-1} \to X^1$) is linear and continuous (see [2] and [3]). The dual operator in the Hilbert space X^0 is denoted by $\mathcal{T}$; the operator $\mathcal{T}$: $X^0 \to X^0$ is continuous. For some $v' \in \mathcal{C}_2^2 \cap C_0 \cap X^1$ and $\chi'_j \in X^0$, let (2.1)–(2.2) hold as indicated in the theorem. It can be verified immediately that $(\mathcal{T}^* h, g)_{X^0} = (h, v')_{X^0}$ $(\forall h \in X^0)$. So $v' = \mathcal{T} g$ in X^0 and hence $v = v'$ in X^0. From (2.3) we obtain that, if v, $v' \in \mathcal{C}_2^2 \cap C_0$ and $v' = v$ in X^0, then $\chi_j = \chi'_j$ $(\forall j)$ in X^0. Thus v and $\chi_1, \cdots, \chi_d$ are determined uniquely in X^0. The theorem has been proved.

The proof of Theorem 2.2 follows from the estimates

$$\|v\|_{X^1} + \|v\|_{C_0} \leqq \|U\|_{\overline{X}^1} + \|U\|_{C_0} \leqq c_1\|g\|_{X^1}, \qquad \|v\|_{W^2} \leqq \|U\|_{\overline{W}^2} \leqq c_2\|g\|_{X^0},$$

which hold for constants $c_i > 0$ common for all g, v, U in the proof of Theorem 2.1, by virtue of known (see [2]–[4]) properties of the operators $\overline{\mathcal{T}}$ and properties of the operation $\mathbf{E}\{\cdot \mid \mathcal{F}_t\}$.

The proof of Theorem 2.3 will be adduced in Section 4.

Proof of Theorem 2.4. Let us consider functions

$$\gamma = \gamma(x,t,\rho,\omega) \in L^2([0,T]\times\Omega,\ \overline{\mathcal{P}},\ \lambda_1\times\mathbf{P},\ C^2(\overline{Q}))\cap L^2([0,T]\times\Omega,\ \overline{\mathcal{P}},\ \lambda_1\times\mathbf{P},\ W_2^2(Q)),$$

which are equal to zero in the case $D \neq \mathbf{R}^n$ for $x \in \partial D$ for a.e. ρ, ω (here $Q = \{(x,t)\}$). Using the estimate [4, p. 523, (149)] for $D \neq \mathbf{R}^n$ and a similar estimate for $D = \mathbf{R}^n$, for a constant $c_1 > 0$ common for all such γ, we obtain the estimate

$$\begin{aligned} \mathbf{E}\int_0^T \|\gamma(x,t,t,\omega)\|^2_{W_2^1(D)}dt &\leqq \mathbf{E}\int_0^T \sup_{t\in[0,T]} \|\gamma(x,t,\rho,\omega)\|^2_{W_2^1(D)}d\rho \\ &\leqq c_1\mathbf{E}\int_0^T \left(\left\|\frac{\partial\gamma}{\partial t}(x,t,\rho,\omega)\right\|^2_{L_2(Q)} + \sum_{i=1}^n \left\|\frac{\partial\gamma}{\partial x_i}(x,t,\rho,\omega)\right\|^2_{L_2(Q)}\right. \\ &\quad \left. + \sum_{i,j=1}^n \left\|\frac{\partial^2\gamma}{\partial x_i\partial x_j}(x,t,\rho,\omega)\right\|^2_{L_2(Q)}\right)d\rho. \end{aligned} \tag{2.9}$$

Obviously this estimate can be extended to all functions $\gamma = \gamma(x,t,\rho,\omega)$ which belong to $C^{2,1}(\overline{Q} \to \mathfrak{L}_2)$, are square-summable in $\overline{Q} \times [0,T] \times \Omega$ together with the corresponding derivatives and are equal to zero for $x \in \partial D$ in the case $D \neq \mathbf{R}^n$. Such are the functions γ_j in the representation (2.6) for $U = \overline{\mathcal{T}}g$, $g \in \mathcal{H}^l$. The right-hand side of the latter inequality in (2.9) under the substitution $\gamma = \gamma_j$ is finite and does not exceed the value

$$c_2\left(\|U\|^2_{\overline{W}^2} + \left\|\frac{\partial U}{\partial t}\right\|^2_{\overline{X}^0}\right) \leqq c_3\|g\|^2_{X^0}, \tag{2.10}$$

where $c_i > 0$ are constants the same for all $g \in \mathcal{H}^l$. Thus

$$\|\gamma_j(x,t,t,\omega)\|_{\mathcal{W}^1} \leqq \sqrt{c_3}\,\|g\|_{X^0}. \tag{2.11}$$

From (1.5)–(1.6) and (2.6) we obtain

$$\begin{aligned} \sum_{j=1}^d \int_s^T \frac{\partial\gamma_j}{\partial s}(x,s,\rho,\omega)\,dw_j(\rho) &= \frac{\partial U}{\partial s}(x,s,\omega) - \mathbf{E}\Big\{\frac{\partial U}{\partial s}(x,s,\omega) \mid \mathcal{F}_s\Big\} \\ &= -A(x,s,\omega)\big[U(x,s,\omega) - \mathbf{E}\{U(x,s,\omega) \mid \mathcal{F}_s\}\big] \\ &= -\sum_{j=1}^d \int_s^T A(x,s,\omega)\gamma_j(x,s,\rho,\omega)\,dw_j(\rho), \end{aligned}$$

for $(x,s) \in Q$ with probability 1. This relation and (2.7) imply that, for a.e. x, t, ω,

$$\chi_j(x,t,\omega) = \gamma_j(x,0,t,\omega) + \int_0^t \frac{\partial\gamma_j}{\partial s}\,(x,s,t,\omega)\,ds = \gamma_j(x,t,t,\omega).$$

By extending the estimate (2.11) from the everywhere dense subset $\mathcal{H}^l$ to X^0, we obtain the assertion of the theorem.

Proof of Theorem 2.5. As is seen from (2.1), the differential

$$d_t v(x,t,\omega) = \tilde{v}(x,t,\omega)\,dt + \chi(x,t,\omega)\,dw(t),$$

where $\tilde{v} = -Av - g \in \mathcal{C}_2^2$, $\chi \in \mathcal{C}_2^2$, exists. We assume that $\tilde{v}(x,t,\omega)$ and $\chi(x,t,\omega)$ are defined on $\mathbf{R}^{n+1} \times \Omega$ and are equal to zero for $(x,t,\omega) \notin \overline{Q} \times \Omega$. For $\varepsilon > 0$ we introduce the functions

$$\begin{aligned}
&\tilde{v}_\varepsilon(x,t,\omega) = \varepsilon^{-1}\int_{t-\varepsilon}^{t} \tilde{v}(x,\rho,\omega)\,d\rho,\\
&\chi_\varepsilon(x,t,\omega) = \varepsilon^{-1}\int_{t-\varepsilon}^{t} \chi(x,\rho,\omega)\,d\rho, \qquad \chi_\varepsilon = \|\chi_1^{(\varepsilon)},\cdots,\chi_d^{(\varepsilon)}\|,\\
&v_\varepsilon(x,t,\omega) = \mathbf{E}U(x,0,\omega) + \int_0^t \tilde{v}_\varepsilon(x,\rho,\omega)\,d\rho + \int_0^t \chi_\varepsilon(x,\rho,\omega)\,dw(\rho).
\end{aligned}$$

We denote $\tau^{x,s}(\varepsilon,\omega) = (T+\varepsilon) \wedge \inf\{t\colon\, y^{x,s}(t,\omega) \notin \overline{D}\}$. From the Itô–Venttsel formula (see [2] and [23]), whose applicability is left without a proof, we see that for a modification of the function v_ε for $(x,s) \in Q$ in the class $\overline{\mathfrak{C}}^{2,0}$ the following relation holds:

$$v_\varepsilon(x,s,\omega) = -\mathbf{E}\left\{\int_s^{\tau^{x,s}(\varepsilon,\omega)} \left[\tilde{v}_\varepsilon + A\tilde{v}_\varepsilon + \sum_{j=1}^{d}\beta_j\frac{\partial\chi_j^{(\varepsilon)}}{\partial x}\right]\big(y^{x,s}(t,\omega),t,\omega\big)dt \mid \mathcal{F}_s\right\} \quad \text{a.s.}$$

Restricting all functions again to $\overline{Q} \times \Omega$, we have, as $\varepsilon \to 0$,

$$\tilde{v}_\varepsilon \longrightarrow -Av - g = \tilde{v}, \qquad Av_\varepsilon \longrightarrow Av, \qquad v_\varepsilon \longrightarrow v, \qquad \beta_j\frac{\partial\chi_j^{(\varepsilon)}}{\partial x} \longrightarrow \beta_j\frac{\partial\chi_j}{\partial x}$$

in the metric of $\mathcal{C}_2^0$, $v_\varepsilon \to v$ in the metric of C_0. In addition $\tau^{x,s}(\varepsilon,\omega) \to \tau^{x,s}(\omega)$ uniformly in $\omega \in \Omega$. Hence we obtain the assertion of the theorem.

Proof of Theorem 2.6. We have $v \in \mathcal{C}_2^{r+2} \cap C_0$, $v(x,0,\omega) = \mathbf{E}U(x,0,\omega)$ for a.e. x,ω, $\mathbf{E}U(x,0,\omega) \in C^{r+2}(\overline{D})$. From this and also from (2.3), with $t = 0$, as well as from the Clark theorem ([21, p. 178]) we obtain successively for $l = 0,1,\cdots,r-1$ for arbitrary i,j and the vector $e_i = \|\delta_{k_i}\|_{k-1}^n$ (where δ_{k_i} is the Kronecker symbol) that the limit of the expression

$$\varepsilon^{-1}\{\mathcal{D}_x^l\chi_j(x+\varepsilon e_i,t,\omega) - \mathcal{D}_x^l\chi_j(x,t,\omega)\},$$

as $\varepsilon \to 0$, exists in X^0 which we denote from now on by $\partial\mathcal{D}_x^l\chi_j/\partial x_i = \mathcal{D}_x^{l+1}\chi_j$. Hence we can approximate $\chi_j(x,t,\omega)$ by functions $\chi_j^{(i)} \in \mathcal{W}^r$ so that $\mathcal{D}_x^l\chi_j^{(i)} \to \mathcal{D}_x^l\chi_j$ in X^0 as $i \to +\infty$, $l = 0,1,\cdots,r$ (we can use averagings in x of the type of [1, p. 48] with a smooth kernel for a.e. t, ω, extending χ_j to $\mathbf{R}^n \times [0,T] \times \Omega$ for $D \neq \mathbf{R}^n$). The completeness of W^r implies the existence of the limit $\chi_j^{(i)}$ in $\mathcal{W}^r$ equivalent to χ_j in X^0. From the inclusion (see [3, p. 61]) $\mathcal{W}^r \subset \mathcal{C}_2^2$ we obtain the required assertion.

3. Forms and properties of dual operators. In addition to the operators $\mathcal{T}$, $\mathcal{G}_j$, B introduced above we shall consider the operators $R = (I+B)^{-1}$, $L = \mathcal{T}R$. The operator R maps a function φ into a solution $g = R\varphi$ of the equation (2.5) connected with the problem (2.1)–(2.2); I is the identity operator. The operator L maps the function φ into the joint solution $v = L\varphi = \mathcal{T}(R\varphi)$ of the equation (2.5) and the problem (2.1)–(2.2).

The symbols $\mathcal{T}^*$, $\mathcal{G}_j^*$, B^*, R^*, L^*, and so on denote the corresponding dual operators in the Hilbert space X^0 (we shall show that the operators R and L are well defined on sets which are everywhere dense in X^0).

For an n-vector $\xi = \|\xi_i\|_{i=1}^n$ we denote $(\nabla, \xi) = \sum_{i=1}^n \partial\xi_i/\partial x_i$.

Below we shall consider initial-boundary value problems of the type of [2, §§ 3.4–4.1] with a boundary condition at $t = 0$. The symbol $\mathcal{X}^0$ will denote the set of processes $h(x, t, \omega)$ which are representatives of some functions in X^0, predictable [2, p. 16] for all x, and taking values in $L_2(D)$ for all t, ω. The symbol $\mathcal{X}^{-1}$ will denote the set of processes h which are representatives of functions (classes) in X^{-1} and representable in the form $h = (\nabla, \xi)$, where $\xi = \|\xi_1, \cdots, \xi_n\|$, $\xi_i \in \mathcal{X}^0$ $(\forall i)$. Solutions of boundary value problems are defined in [2] for free terms in $\mathcal{X}^k$. It is known [24, Chap. 3] that in every equivalence class of X^{-1}, X^0 there are representatives of $\mathcal{X}^{-1}$, $\mathcal{X}^0$, respectively. Therefore, we can (and shall) understand by a solution of boundary value problems of the type of [2] with an initial condition at $t = 0$ for free terms in X^k, $k = -1, 0$, an extension to these Hilbert spaces of continuous operators (using suitable theorems of [2]) which map free terms of boundary value problems into solutions in $X^1 \cap C_0$. Then a boundary condition of the form $g(x, t, \omega)|_{(x,t)\in\partial_0 Q} = 0$ is said to be satisfied if $g \in X^1 \cap C_0$ and $g(x, 0, \omega) = 0$ for a.e. x, ω.

THEOREM 3.1. *The operators $\mathcal{G}_j^*$: $X_0 \to X_1$ are continuous and have the form $\mathcal{G}_j^* h = q$, where the function $q \in X^1 \cap C_0$ satisfies the boundary value problem*

$$d_t q(x, t, \omega) = A^*(x, t, \omega) q(x, t, \omega)\, dt + h(x, t, \omega)\, dw_j(t), \tag{3.1}$$

$$q(x, t, \omega)|_{(x,t)\in\partial_0 Q} = 0. \tag{3.2}$$

THEOREM 3.2. *The operator B^*: $X^0 \to X^0$ is continuous and has the form $B^* h = z$, where the function z satisfies the boundary value problem*

$$d_t z(x, t, \omega) = A^*(x, t, \omega) z(x, t, \omega)\, dt + \sum_{j=1}^{d} \big(\nabla, \beta_j(x, t, \omega) h(x, t, \omega)\big) dw_j(t), \tag{3.3}$$

$$z(x, t, \omega)|_{(x,t)\in\partial_0 Q} = 0. \tag{3.4}$$

For $h \in \mathcal{X}^1$ the solution $z = B^* h \in X^1 \cap C_0$ is understood in the sense of [2], *for $h \in X^0$, $h \notin \mathcal{X}^1$, the solution is the limit in X^0 of a sequence $B^* h_i$, where $h_i \in \mathcal{X}^1$ and $\|h_i - h\|_{X^0} \to 0$ as $i \to +\infty$.*

The theorem stated above contains the assertion of existence of a "generalized" solution in the class X^0 (or of the possibility of defining a solution as the corresponding limit in this space) for a coefficient belonging to the class X^{-1} of the stochastic differential in the free term of the equation. This assertion is apparently new for the theory of partial Itô equations.

THEOREM 3.3. *For $d < d_0$, the operator R^*: $X^0 \to X^0$ is determined uniquely and the operator R^*: $X^1 \to X^1$ is continuous. For $\pi \in X^1$, this operator has the form*

$R^*\pi = h$, where $h = \pi - z$ and the function $z \in X^1 \cap C_0$ is a solution of the boundary value problem

$$
\begin{aligned}
d_t z(x,t,\omega) &= A^*(x,t,\omega) z(x,t,\omega)\,dt \\
&\quad + \sum_{j=1}^{d} \big(\nabla,\ \beta_j(x,t,\omega)\big[\pi(x,t,\omega) - z(x,t,\omega)\big]\big)\,dw_j(t),
\end{aligned} \tag{3.5}
$$

$$
z(x,t,\omega)|_{(x,t)\in\partial_0 Q} = 0. \tag{3.6}
$$

THEOREM 3.4. *For $d < d_0$, the operator $L^*\colon X^1 \to X^1$ is continuous and has the form $L^*\xi = h$, where the function $h \in X^1 \cap C_0$ is a solution of the boundary value problem*

$$
\begin{aligned}
d_t h(x,t,\omega) &= \big[A^*(x,t,\omega) h(x,t,\omega) + \xi(x,t,\omega)\big]\,dt \\
&\quad - \sum_{j=1}^{d} \big(\nabla,\ \beta_j(x,t,\omega) h(x,t,\omega)\big)\,dw_j(t),
\end{aligned} \tag{3.7}
$$

$$
h(x,t,\omega)|_{(x,t)\in\partial_0 Q} = 0. \tag{3.8}
$$

Let us note that (3.1) and (3.3) are superparabolic [2] Itô equations, and (3.5) and (3.7) are superparabolic for $d < d_0$ and parabolic for $d = d_0$.

Proof of Theorem 3.1. First let f and β be nonrandom.

Suppose that $g \in \mathcal{H}^l$ is an arbitrary function and the functions $\hat g_j \in C(\overline{Q} \to \mathfrak{L}_2)$ are determined by the Clark theorem [21, p. 178] from the representation

$$
g(x,t,\omega) = \mathbf{E} g(x,t,\omega) + \sum_{j=1}^{d} \int_0^t \hat g_j(x,t,\rho,\omega)\,dw_j(\rho); \tag{3.9}
$$

the functions $u(x,t,s,\omega) \in \mathcal{C}_2^2(s)$ and $\gamma_j(x,t,\rho,\omega) \in C^{2,1}(\overline{Q} \to \mathfrak{L}_2)$ for $U = \overline{\mathcal{T}} g \in \overline{\mathfrak{C}}^{2,1}$ are defined in the same way as in the proof of Theorem 2.1. We have

$$
\begin{aligned}
\sum_{j=1}^{d} \int_0^t \frac{\partial \gamma_j}{\partial t}(x,t,\rho,\omega)\,dw_j(\rho) &= \mathbf{E}\Big\{\frac{\partial U}{\partial t}(x,t,\omega) \mid \mathcal{F}_t\Big\} - \mathbf{E}\frac{\partial U}{\partial t}(x,t,\omega) \\
&= -\Big[A(x,t) u(x,t,t,\omega) + \mathbf{E}\{g(x,t,\omega) \mid \mathcal{F}_t\} \\
&\quad - A(x,t) u(x,t,0,\omega) - \mathbf{E} g(x,t,\omega)\Big] \\
&= -\sum_{j=1}^{d} \int_0^t \big[A(x,t)\gamma_j(x,t,\rho,\omega) + \hat g_j(x,t,\rho,\omega)\big]\,dw_j(\rho).
\end{aligned} \tag{3.10}
$$

Let $G(x,y,t,s)$ be Green's function of the boundary value problem (1.5), (1.6) with the nonrandom operator $A(x,t,\omega) = A(x,t)$; then from (3.10) and the condition $\gamma_j(x,t,\rho,\omega)|_{(x,t)\in\partial_0 Q} = 0$ for a.e. ρ, ω we obtain that, for a.e. ρ, ω,

$$
\gamma_j(x,t,\rho,\omega) = \int_D dy \int_t^T G(x,y,t,s)\hat g_j(y,s,\rho,\omega)\,ds.
$$

For an arbitrary function $h \in \mathcal{H}^l$ we have

$$(\mathcal{G}_j g, h)_{X^0} = \mathbf{E}\int_0^T dt \int_D dx \left\{ \left[\int_D dy \int_t^T G(x,y,t,s)\hat{g}_j(y,s,t,\omega)\, ds \right] h(x,s,\omega) \right\}$$
$$= \mathbf{E}\int_0^T ds \int_D dy \int_0^s dt\, \hat{g}_j(y,s,t,\omega) \int_D G(x,y,t,s,)h(x,t,\omega)\, ds = (g, \mathcal{G}_j^* h)_{X^0}.$$

In view of (3.9) and the fact that g is arbitrary this means that

$$(\mathcal{G}_j^* h)(y,s,\omega) = \int_0^s dw_j(t) \int_D G(x,y,t,s)h(x,t,\omega)\, dx.$$

From this relation we obtain (3.1), (3.2) and the form of $\mathcal{G}_j^*$ for nonrandom f, β.

Now let f, β be random. Consider the functions $f_0(x,t) = \mathbf{E} f(x,t,\omega)$, $\beta_0(x,t) = \mathbf{E}\beta(x,t,\omega)$. Let A_0, A_0^*, $\mathcal{T}_0$, $\mathcal{G}_{j,0}^*$ denote the operators corresponding to the function f_0, β_0, which are defined like the A, A^*, $\mathcal{T}$, $\mathcal{G}_j$ are defined for the functions f, β. We introduce the operator $\mathfrak{A} = (A_0 - A)\mathcal{T}_0$: for $g_0 \in X^0$ we have $\mathfrak{A} g_0 = (A_0 - A)v_0$, where $v_0 = \mathcal{T}_0 g_0$. The operators $\mathfrak{A}$: $X^{-1} \to X^1$ and $\mathfrak{A}$: $X^0 \to X^0$ are continuous by Theorem 2.2.

It can be verified immediately that, for $g = g_0 + \mathfrak{A} g_0$, we have $\mathcal{T} g = \mathcal{T}_0 g_0$ and $\mathcal{G}_j g = \mathcal{G}_{j,0} g_0$. This means that $\mathcal{G}_j = \mathcal{G}_{j,0}(I + \mathfrak{A})^{-1}$ and the dual operator in X^0 has the form $\mathcal{G}_j^* = (I + \mathfrak{A}^*)^{-1} \mathcal{G}_{j,0}^*$.

Obviously $\mathfrak{A}^* = \mathcal{T}_0^*(A_0^* - A^*)$. The form of $\mathcal{T}^*$ (and analogously of $\mathcal{T}_0^*$) was established in §2 by formulas (2.8). The operators $\mathcal{T}^*$ and $\mathcal{T}_0^*$ map X^{-1} continuously (see [2]–[4]) into X^1 and C_0. The operators A^* and A_0^* map X^1 continuously into X^{-1}.

For $q \in X^1$ we have $z = \mathfrak{A}^* q \in X^1 \cap C_0$, and $z = z(x,t,\omega)$ satisfies the boundary value problem $dz/dt = A_0^* z + (A_0^* - A^*)q$, $z|_{(x,t)\in\partial_0 Q} = 0$ in Q.

Let us find the form of $q = (I + \mathfrak{A}^*)^{-1}\eta$ for $\eta \in X^1$. We have $q + \mathfrak{A}^* q = \eta$. We denote $z = \mathfrak{A}^* q$; then $q = \eta - z$ and $z = \mathfrak{A}^*(\eta - z)$. By the assertion proved above the function $z = z(x,t,\omega)$ satisfies the boundary value problem

$$\frac{dz}{dt} = A_0^* z + (A_0^* - A^*)(\eta - z) = A^* z + (A_0^* - A^*)\eta, \qquad z|_{(x,t)\in\partial_0 Q} = 0. \tag{3.11}$$

Thus, $z = \eta - q \in X^1 \cap C_0$ and $q \in X^1$.

Let us find the form of $q = (I + \mathfrak{A}^*)^{-1}\eta$ for $\eta = \mathcal{G}_{j,0}^* h$, where $h \in X^0$. By Theorems 3.4.8 and 4.1.1 of [2] the function $\eta = \eta(x,t,\omega) \in X^1 \cap C^0$ and satisfies, by the proof, the boundary value problem $d_t\eta = A_0^*\eta\, dt + h\, dw_j(t)$, $\eta|_{(x,t)\in\partial_0 Q} = 0$. Moreover, $q = \eta - z$, where the function $z = z(x,t,\omega)$ satisfies a boundary value problem of the form (3.11). So $q \in X^1 \cap C_0$. From the formulas for $d_t\eta$, and $d_t z = (dz/dt)\, dt$ we find $d_t q = d_t\eta - d_t z$ and thus we obtain (3.1). Condition (3.2) is satisfied since the analogous conditions hold for z and η. Continuity of the operator $\mathcal{G}_j^*$: $X^0 \to X^1$ (and even continuity of the operator $\mathcal{G}_j^*$: $X^0 \to C_0$) follows from Theorems 3.4.8 and 4.1.1 of [2]. The theorem has been proved.

Proof of Theorem 3.2. By Theorem 2.4, for $g \in X^0$, $\chi_j = \mathcal{G}_j g \in X^1$. For $h \in X^1$ we have

$$(Bg, h)_{X^0} = \sum_{j=1}^d (\chi_j, (\nabla, \beta_j h))_{X^0} = \left(g, \sum_{j=1}^d \mathcal{G}_j^*(\nabla, \beta_j h) \right)_{X^0}.$$

From this relation and the linearity of the problem (3.1), (3.2) we obtain (3.3), (3.4). Theorems 3.4.8 and 4.1.1 of [2] imply continuity of the operators $B^*\colon X^1 \to X^1$ and $B^*\colon X^1 \to C_0$. Continuity of the operator $B\colon X^0 \to X^0$ proved in Theorem 2.4 implies continuity of the operator $B^*\colon X^0 \to X^0$.

Proof of Theorem 3.3. If $h = R^*\pi$, then $h = \pi - z$, where $z = B^*h = B^*(\pi - z)$. By substituting the value $h = \pi - z$ into (3.3), (3.4) we obtain formulas (3.5), (3.6). Continuity of the operator $R^*\colon X^1 \to X^1$ (and thus uniqueness of the operator $R^*\colon X^0 \to X^0$) follows from Theorems 3.4.8 and 4.1.1 of [2].

Proof of Theorem 3.4. Let $h = R^*\pi$ and $z = B^*h$; then $h = \pi - z$. Let $\pi = \mathcal{T}^*\xi$ be determined from the problem (2.8), where $\xi \in X^{-1}$. Using (2.8) and (3.3), (3.4) we obtain the expression for $d_t h = d_t\pi - d_t z$ or (3.7), as well as condition (3.8). Continuity of the operator $L^*\colon X^{-1} \to X^1$ follows from the form of $h = L^*\xi$ and Theorems 3.4.8 and 4.1.1 of [2].

4. Solvability of (2.5).

THEOREM 4.1. *Let $d < d_0$. Then the operator $R\colon X^0 \to X^0$ is unique and well-defined on some everywhere dense set $\mathcal{D}(R) \subset X^0$ in X^0 (and in X^{-1}) (that is, (2.5) has at most one solution $g \in X^0$ for any $\varphi \in X^0$ and, in addition, the set of those $\varphi \in X^0$, for which the equation is solvable with respect to $g \in X^0$, is everywhere dense in X^0). The operators R and L defined on $\mathcal{D}(R)$ can be extended from this set to operators defined on X^{-1} so that the operators $R\colon X^{-1} \to X^{-1}$, $L\colon X^{-1} \to X^1$, and $L\colon X^{-1} \to C_0$ are continuous.*

Let $\mathcal{C}_*$ denote the set of all $\varphi \in \mathcal{C}_2^0 \cap X^0$ such that up to equivalence (in X^0) $\varphi = g + Bg$ for some $g \in \mathcal{H}^l$, where $l > r$, the integer number $r \geqq 0$, and the number l were introduced in §1. We recall that such φ occured in the statement of Theorem 2.6, which asserts that for these φ with $r > n/2 + 2$ there exists a modification in the class $\mathcal{C}_2^0$ (and hence in the class $\mathcal{C}_*$) and moreover for this modification the value of the functional (1.3) coincides with $L\varphi$.

THEOREM 4.2. *For $d > d_0$ and $r > n/2 + 2$, the set $\mathcal{C}_*$ is everywhere dense in X^0.*

Proof of Theorem 4.1. The operator R^* is defined on the set X^1 which is everywhere dense in X^0 and maps X^1 continuously into X^1. Hence the operator $I + B^*$ inverse to it has a set of values everywhere dense in X^0 and a kernel consisting only of zero. Obviously, the operator $I + B$ has the same properties. Thus the operator R is determined uniquely and has a domain that is everywhere dense in X^0 and is denoted by $\mathcal{D}(R)$. Since X^0 is everywhere dense in X^{-1}, $\mathcal{D}(R)$ is everywhere dense in X^{-1}.

It remains to prove the assertion concerning the extension of the operators to X^{-1}. For $k = 0, \pm 1, \pm 2$, $u = u(x,t,\omega) \in X^0$, the symbol $\Lambda^k u$ will denote the function in X^{-k} obtained by the application, for a.e. t, ω, of the operator Λ^k introduced in §1 in the definition of the spaces H^k to the function $u(\cdot,t,\omega) \in L_2(D)$. In view of Theorem 3.3 we have, for some constant $c > 0$ and for any $\varphi \in \mathcal{D}(R)$, $h \in X^0$,

$$\begin{aligned}(R\varphi, h)_{X^{-1}} &= \left(\Lambda^{-1}R\varphi, \Lambda^{-1}h\right)_{X^0} = \left(R\varphi, \Lambda^{-2}h\right)_{X^0} = \left(\Lambda^{-1}\varphi, \Lambda R^*\Lambda^{-2}h\right)_{X^0} \\ &\leqq \|\varphi\|_{X^1}\left\|R^*\Lambda^{-2}h\right\|_{X^{-1}} \leqq c\|\varphi\|_{X^{-1}}\left\|\Lambda^{-2}h\right\|_{X^1} \leqq c\|\varphi\|_{X^{-1}}\|h\|_{X^{-1}}.\end{aligned}$$

This inequality implies the existence of a continuous extension of the operator R to X^{-1}. The corresponding assertion of the theorem for the operator $L = \mathcal{T}R$ follows from Theorem 2.2. The theorem has been proved.

Proof of Theorem 4.2. For an arbitrary number $\varepsilon > 0$ and for $\varphi \in X^0$, it is required to find $\varphi_0 \in \mathcal{C}_*$ such that $\|\varphi - \varphi_0\|_{X^0} < \varepsilon$. By Theorem 4.1 there exists $\varphi' \in X^0$ such that $\varphi' = g' + Bg'$ for some $g' \in X^0$ and $\|\varphi - \varphi'\|_{X^0} < \varepsilon/2$. The norm $\|I + B\|$ of the operator $(I + B)$: $X^0 \to X^0$ is positive since this operator has an image which is everywhere dense in X^0. For $g' \in R\varphi$, there exists $g'' \in \mathcal{H}^l$ such that $\|g' - g''\|_{X^0} < \varepsilon\|I + B\|^{-1}/2$. By Theorem 2.6, the function $\varphi_0 = g'' + Bg''$ has a modification in the class $\mathcal{C}_2^0$. This function $\varphi \in \mathcal{C}_2^0$ is the one required since

$$\|\varphi - \varphi_0\|_{X^0} \leqq \|\varphi - \varphi'\|_{X^0} + \|\varphi' - \varphi_0\|_{X^0} < \varepsilon/2 + \|g' - g''\|_{X^0}\|I + B\| < \varepsilon.$$

The theorem has been proved.

Proof of Theorem 2.3. By virtue of Theorem 3.1, we have, for some constant $c > 0$ and for any $g \in X^0$, $h \in X^0$,

$$\begin{aligned}(\mathcal{G}_j g, h)_{X^0} &= (g, \mathcal{G}_j^* h)_{X^0} = \left(\Lambda^{-1} g, \Lambda \mathcal{G}_j^* h\right)_{X^0} \leqq \|\Lambda^{-1} g\|_{X^0} \left\|\Lambda \mathcal{G}_j^* h\right\|_{X^0} \\ &= \|g\|_{X^{-1}} \left\|\mathcal{G}_j^* h\right\|_{X^1} \leqq c\|g\|_{X^{-1}} \|h\|_{X^0}.\end{aligned}$$

This inequality implies the assertion of the theorem.

5. Representation of functionals of Itô processes in the form of solutions of boundary value problems. Let us adduce some sufficient conditions for the functional (1.3) to coincide, for a given φ, with a solution of the problem (2.1), (2.2) and (2.5).

THEOREM 5.1. *Let $D = \mathbf{R}^n$ or $D \neq \mathbf{R}^n$, $d < d_0$; let the function $\beta(x,t,\omega) = \beta(t,\omega)$ not depend on x, the function $f \in \mathcal{C}_2^2$, and let at least one of the following conditions hold*:

a) *the function $f(x,t,\omega) = f(t,\omega)$ does not depend on x and the function $\varphi \in \mathcal{C}_2^0 \cap X^0$*;

b) *$n = 1$ and the function $\varphi \in \mathcal{C}_2^0 \cap X^0$*;

c) *$n = 1$ and $\varphi = \varphi(x,t)$ is a nonrandom Borel measurable function of $L_2(Q)$.*

Then the value $v(x,s,\omega)$ of the functional (1.3) *as a function of (x,s,ω) belongs to $X^1 \cap C_0$ and coincides with $L\varphi$ as a function in X^0 and in C_0 (i.e., is a solution of the problem* (2.1), (2.2), and (2.5)).

COROLLARY 5.1. *Under the assumptions of Theorems* 4.1 *and* 5.1 *the estimates*

$$\|v\|_{C_0} \leqq c_1 \|\varphi\|_{X^{-1}} \leqq c_2 \|\varphi\|_{X^0}, \qquad \|v\|_{X^1} \leqq c_3 \|\varphi\|_{X^{-1}} \leqq c_4 \|\varphi\|_{X^0},$$

hold for the functional (1.3), *where the constants $c_i > 0$ depend only on n, d, d_0, Q, f, β, and, more precisely, as is seen from the proofs of* §3 *and* [2], *only on n, d, d_0, Q, and the values*

$$\delta = \inf_{x,t,\omega} \operatorname{Det} \tilde{\beta}(x,t,\omega)\tilde{\beta}(x,t,\omega)^T, \qquad K_1 = \sup_{x,t,\omega} \left|f(x,t,\omega)\right|,$$

$$K_2 = \sup_{x,t,\omega} \left|\beta(x,t,\omega)\right|, \qquad K_3^i = \sup_{x,t,\omega} \left|\frac{\partial \beta}{\partial x_i}(x,t,\omega)\right|$$

(*cf. estimates in* [1, §§ II.2–II.3]).

Proof of Theorem 5.1. Let assumptions a) or b) hold. For a function $\eta(x) \in L_2(\mathbf{R}^n)$ the symbol $(\eta)_\varepsilon$ will denote its averaging (convolution) with the kernel of the Sobolev averaging $\zeta(x/\varepsilon)\varepsilon^{-n}$. Here the function $\zeta(x) = 0$ for $|x| \geqq 1$, $\zeta(x) =$

$\varkappa_n \exp\{|x|^2(|x|^2-1)\}$ for $|x|<1$; $\varkappa_n$ is a normalizing factor such that $\int_{\mathbf{R}^n}\zeta(x)\,dx=1$. If $\eta\in H^{-1}$ and $\eta=\partial\xi/\partial x_j$, where $\xi=L_2(\mathbf{R}^n)$, $j\in\{1,\cdots,n\}$, then we assume that

$$(\eta)_\varepsilon(x)=-\varepsilon^{-n-1}\int_{\mathbf{R}^n}\frac{\partial\zeta}{\partial x_j}\Big(\frac{y-x}{\varepsilon}\Big)\xi(y)\,dy.$$

Let $D=\mathbf{R}^n$. For functions $u\in X^0$ or $u\in X^{-1}$, the symbol $(u)_\varepsilon$ denotes a function in $\mathcal{C}_2^0$ coinciding with $(u(\cdot,t,\omega))_\varepsilon$ for all t,ω such that $u(\cdot,t,\omega)\in H^0=L_2(\mathbf{R}^n)$ or $u(\cdot,t,\omega)\in H^{-1}$, respectively (i.e., for a.e. t, ω).

Let $D\neq\mathbf{R}^n$. In this case, functions defined on $\overline{Q}\times\Omega$ are assumed to be extended to $\mathbf{R}^n\times[0,T]\times\Omega$, and the operation $(\cdot)_\varepsilon$ is applied to them according to the rule indicated above.

Everywhere in the proof of this theorem, $\mathcal{C}_m^2$ will be the space $\mathcal{C}_2^m=L^2([0,T]\times\Omega,\ \overline{\mathcal{P}},\ \lambda_1\times\mathbf{P},\ C^m(\mathbf{R}^n))$. So, for $m=0,1,2,\cdots$ and $u\in X^0$ or $u\in X^{-1}$, we have $(u)_\varepsilon\in\mathcal{C}_2^m$ for the spaces X^0, X^{-1} defined for $D=\mathbf{R}^n$ as well as $D\neq\mathbf{R}^n$.

For $D\neq\mathbf{R}^n$ we denote by D_ε a region with a C^2-smooth boundary which contains the union of 2ε-neighborhoods for all $x\in D$ and which itself is contained in the union of 3ε-neighborhoods for all $x\in D$. The symbol $\tau_\varepsilon^{x,s}(\omega)$ denotes the random time $T\wedge\inf\{t\colon\ y^{x,s}(t,\omega)\notin\overline{D}_\varepsilon\}$. For $D=\mathbf{R}^n$ we assume that $D_\varepsilon=D=\mathbf{R}^n$, $\tau_\varepsilon^{x,s}(\omega)\equiv\tau^{x,s}(\omega)\equiv T$.

Let $D=\mathbf{R}^n$ or $D\neq\mathbf{R}^n$, $g=R\varphi\in X^{-1}$, $v=L\varphi$, $\chi=\mathcal{G}g$, $\chi_j=\mathcal{G}_jg$. Then $v\in X^1\cap C_0$, $\chi_j\in X^0$ and, for all s,x, we have

$$(v)_\varepsilon(x,s,\omega)=\int_s^T(Av+g)_\varepsilon(x,t,\omega)\,dt-\int_s^T(\chi)_\varepsilon(x,t,\omega)\,d\omega(t),$$

$$(v)_\varepsilon(x,t,\omega)|_{x\in\partial D_\varepsilon}=0\qquad\text{in the case}\quad D\neq\mathbf{R}^n,\quad v_\varepsilon(x,T,\omega)=0$$

with probability 1.

We introduce the functions $\Delta_\varepsilon=(Av)_\varepsilon-A(v)_\varepsilon$, $\Phi_\varepsilon=\Delta_\varepsilon+(\varphi)_\varepsilon$. These functions belong to the class $\mathcal{C}_2^2$. The function $(v)_\varepsilon\in\mathcal{C}_2^4$ is a solution of the problem of the form (2.1), (2.2) with the free term $(g)_\varepsilon+\Delta_\varepsilon\in\mathcal{C}_2^2\cap X^{-1}$ in the cylinder $D_\varepsilon\times(0,T)$. By Theorem 2.5, for any $s\in[0,T]$ for a.e. $(s,\omega)\in D_\varepsilon\times\Omega$ and for a.e. $(s,\omega)\in[0,T]\times\Omega$, we have

$$\begin{aligned}(v)_\varepsilon(x,s,\omega)&=\mathbf{E}\Big\{\int_s^{\tau_\varepsilon^{x,s}(\omega)}\Phi_\varepsilon\big[y^{x,s}(t,\omega),t,\omega\big]dt\mid\mathcal{F}_s\Big\}\\&=\mathbf{E}\Big\{\int_s^{\tau^{x,s}(\omega)}\Phi_\varepsilon\big[y^{x,s}(t,\omega),t,\omega\big]dt\mid\mathcal{F}_s\Big\}\\&\quad+\mathbf{E}\Big\{\int_{\tau^{x,s}(\omega)}^{\tau_\varepsilon^{x,s}(\omega)}\Phi_\varepsilon\big[y^{x,s}(t,\omega),t,\omega\big]dt\mid\mathcal{F}_s\Big\}\end{aligned}\tag{5.1}$$

for any $x\in D_\varepsilon$.

Let us estimate Δ_ε. For a.e. t, ω we have $v(\cdot,t,\omega)\in H^1$ and

$$\Delta_\varepsilon(x,t,\omega)=\varepsilon^{-n}\int_D\frac{\partial v}{\partial x}(y,t,\omega)\big[f(y,t,\omega)-f(x,t,\omega)\big]\zeta\Big(\frac{x-y}{\varepsilon}\Big)dy.$$

Under assumption a) this value is equal to zero.

The Hölder inequality and the inequality $\zeta(x)^2 < \varkappa_n\zeta(x)$ imply that

$$\begin{aligned}
&|\Delta_\varepsilon(x,t,\omega)| \\
&\leqq c_1\varepsilon^{-n}\Big(\sup_{|x-y|\leqq\varepsilon}|f(x,t,\omega)-f(y,t,\omega)|\Big)\|v(\cdot,t,\omega)\|_{H^1}\Big(\int_{\mathbf{R}^n}\zeta\Big(\frac{x-y}{\varepsilon}\Big)^2dy\Big)^{1/2} \\
&\leqq c_1\varepsilon^{-n}c_2\varepsilon\Big(\sup_{x,t,\omega}\Big|\frac{\partial f}{\partial x}(x,t,\omega)\Big|\Big)\|v(\cdot,t,\omega)\|_{H^1}\varepsilon^{n/2} = \xi_1(t,\omega)\varepsilon^{n/2-n+1}
\end{aligned}$$

for a.e. t,ω, where $c_i > 0$ are constants and $\xi_1(t,\omega)$ is some function in $L^2([0,T]\times\Omega,\ \overline{\mathcal{P}},\ \lambda_1\times\mathbf{P},\ \mathbf{R})$.

Let $\varepsilon\to 0$. For the function

$$\xi_2(t,\omega) = 2\sup_{x\in D}|\varphi(x,t,\omega)| \in L^2\big([0,T]\times\Omega,\ \overline{\mathcal{P}},\ \lambda_1\times\mathbf{P},\ \mathbf{R}\big),$$

we have $|(\varphi)_\varepsilon(x,t,\omega)| + |\varphi(x,t,\omega)| \leqq \xi_2(t,\omega)$ for a.e. t, ω. Moreover, for a.e. t, ω, $(\varphi)_\varepsilon(x,t,\omega)\to\varphi(x,t,\omega)$ for any $x\in D$. We have also $\Delta_\varepsilon\to 0$ in the metric of C_2^0 for $n=1$. Thus,

$$\mathbf{E}\int_0^T\Big|(\varphi)_\varepsilon\big[y^{x,s}(t,\omega),t,\omega\big]-\varphi\big[y^{x,s}(t,\omega),t,\omega\big]\Big|^2dt\longrightarrow 0,$$

$$\mathbf{E}\int_0^T\Big|\Delta_\varepsilon\big[y^{x,s}(t,\omega),t,\omega\big]\Big|^2dt\longrightarrow 0.$$

From these relations and the Lebesgue theorem we see that the first term in the right-hand side of equality (5.1) tends, in the metric of $L^2(\Omega,\mathcal{F},\mathbf{P},\mathbf{R})$, to the right-hand side of equality (1.3). Moreover, the left-hand side of (5.1) tends to $v = L\varphi$ in the metrics of C_0 and X^0 as a function of (x,s,ω).

To complete the proof of the theorem for the case of assumptions a) and b) we prove that the second term in the right-hand side of (5.1) tends to zero in the metric of $L^2(\Omega,\mathcal{F},\mathbf{P},\mathbf{R})$. Obviously this term is equal to zero in case $D=\mathbf{R}^n$. For $D\neq\mathbf{R}^n$ and $n=1$, we obtain

$$\Phi_\varepsilon\big[y^{x,s}(t,\omega),t,\omega\big]\leqq\xi_1(t,\omega)\varepsilon^{1/2}+\xi_2(t,\omega)$$

for a.e. t,ω. Moreover, $\tau_\varepsilon^{x,s}(\omega)\downarrow\tau^{x,s}(\omega)$ a.s. since $\tau^{x,s}(\omega) = T\wedge\inf\{t\colon y^{x,s}(t,\omega)\notin\overline{D}\}$ and for a.e. ω there exists $\bar\theta=\bar\theta(\omega)>0$ such that $y^{x,s}(t,\omega)\in\mathbf{R}^n\backslash\overline{D}$ for $\tau^{x,s}(\omega)<T$, $t=\tau^{x,s}(\omega)+\theta$, $\theta\in(0,\bar\theta(\omega)]$. Hence we obtain the required assertion for assumptions a) and b).

Let assumption c) hold. We introduce the operator $\widetilde{L}$ defined on $L_2(Q)$, mapping functions $\varphi\in L_2(Q)$ into values $\tilde v=\widetilde{L}\varphi$ of the functional (1.3) regarded as functions of (x,s,ω).

Assume $\varphi\in L_2(Q)$, $\tilde v=\widetilde{L}\varphi$, the sequence $\{\varphi_i\}_{i=1}^{+\infty}\subset C(\overline{Q})$, $\tilde v_i=\widetilde{L}\varphi_i$, and $\varphi_i\to\varphi$ in the metric of $L_2(Q)$ as $i\to+\infty$. By the above proof, $\widetilde{L}\varphi_i = L\varphi_i\in C_0$. Theorems II.2.4 and II.3.4 of [1] imply that, for some constant $c>0$,

$$\sup_{t\in[0,T]}\mathbf{E}\|\tilde v(x,t,\omega)-v_i(x,t,\omega)\|_{H^0}^2\leqq c\|\varphi-\varphi_i\|_{L_2(Q)}.$$

Completeness of the space C_0 implies that $\tilde v_i\to v$ in C_0 as $t\to+\infty$ and $\tilde v\in C_0$. We have $\widetilde{L}\varphi_i = L\varphi_i$ and, by Theorem 4.1, $\widetilde{L}\varphi_i = L\varphi_i\to L\varphi$ in C_0 as $i\to+\infty$. Hence $\tilde v=\widetilde{L}\varphi=L\varphi$.

Theorem 5.1 has been proved.

6. On distributions of Itô processes. Let $D = \mathbf{R}^n$ or $D \neq \mathbf{R}^n$, $d < d_0$ and let $p_0(x) \in L_2(D)$ be some nonrandom functions. We consider in the cylinder Q the boundary value problem

$$d_t p(x,t,\omega) = A^*(x,t,\omega)p(x,t,\omega)\,dt - \sum_{j=1}^{d} \big(\nabla, \beta_j(x,t,\omega)p(x,t,\omega)\big)dw_j(t), \tag{6.1}$$

$$p(x,0,\omega) = p_0(x), \qquad p(x,t,\omega)|_{x\in\partial D} = 0. \tag{6.2}$$

The boundary condition on ∂D in (6.2) is not considered for $D = \mathbf{R}^n$.

Equation (6.1) is a superparabolic Itô equation [2]. A solution of the problem (6.1)–(6.2) is understood to be analogous to [2]; this problem has a solution $p \in X^1 \cap C_0$. The boundary conditions (6.2) for $p \in X^1 \cap C_0$ are said to be satisfied if $p(x,0,\omega) = p_0(x)$ for a.e. x, ω.

LEMMA 6.1. *For $\varphi \in X^0$ and $s \in [0,T]$, the equality*

$$\int_D p(x,s,\omega)v(x,s,\omega)\,dx = \mathbf{E}\bigg\{\int_s^T dt \int_D p(x,t,\omega)\varphi(x,t,\omega)\,dx \mid \mathcal{F}_s\bigg\} \tag{6.3}$$

holds with probability 1. *Here $v = L\varphi \in X^1 \cap C_0$ is a solution of the problem* (2.1), (2.2), (2.5).

Let, in (1.1), (1.2), $s = 0$ and let $x = x(\omega)$ be a random n-vector. We assume that $\mathbf{E}|x(\omega)|^2 < +\infty$, $x(\omega) \in D$ a.s., the vector $x(\omega)$ does not depend on $W(t)$ for any $t \geqq 0$ and has a probability density $p_0(x) \in L_2(D)$. Let $y^{x(\omega),0}(t,\omega)$ be the corresponding solution of equations (1.1), (1.2) and let the random time $\tau^{x(\omega),0}(\omega) = T \wedge \inf\{t\colon\ y^{x(\omega),0}(t,\omega) \notin \overline{D}\}$. The symbol $I_\tau(t,\omega)$ denotes the indicator function of the event $\{\tau^{x(\omega),0}(\omega) \geq t\}$. For $D = \mathbf{R}^n$ we have $\tau^{x(\omega),0}(\omega) = T$, $I_\tau = (\tau,\omega) \equiv 1$, for $0 \leqq t \leqq T$.

THEOREM 6.1. *Let $\varphi \in \overline{\mathfrak{C}} \cap C_2^0 \cap X^0$, and let the assumptions of Theorem* 5.1 *hold. Then for a.e. $(t,\omega) \in [0,T] \times \Omega$ (and even for any $t \in [0,T]$ almost surely if $D = \mathbf{R}^n$) the following equality holds*:

$$\mathbf{E}\big\{I_\tau(t,\omega)\varphi\big[y^{x(\omega),0}(t,\omega),t,\omega\big] \mid \mathcal{F}_t\big\} = \int_D p(x,t,\omega)\varphi(x,t,\omega)\,dx. \tag{6.4}$$

This theorem establishes the distribution of the process $y^{x(\omega),0}(t,\omega)$ (broken off at the exit of $\overline{D}$ if $D \neq \mathbf{R}^n$); $\mathbf{E}p(x,t,\omega)$ is the distribution density of the process. A close result is proved in [2, Thm. 5.3.1], where equality (6.4) is obtained for $D = \mathbf{R}^n$ and coefficients f, β of general form (no restrictive assumptions of Theorem 5.1 are required). Moreover, in [2] another method of the proof is used, and equality (6.4) is obtained only for nonrandom φ and $D = \mathbf{R}^n$, which is essential. Theorem 5.3.1 of [2] establishes the distribution of the Itô process $y^{x(\omega),0}(t,\omega)$; therefore with its help we can obtain the following analogue of Theorem 5.1 (less strong, however, for functions f and β of general form).

THEOREM 6.2. *Let $D = \mathbf{R}^n$, $d < d_0$, and let the function $\varphi(x,t) \in C(\overline{Q}) \cap L_2(Q)$ be nonrandom. Then*

$$\mathbf{E}\int_0^T \varphi\big[y^{x(\omega),0}(t,\omega),t\big]dt = \int_{\mathbf{R}^n} p_0(x)\bar{v}(x,0)\,dx = \mathbf{E}\bar{v}\big[x(\omega),0\big]. \tag{6.5}$$

Here $\bar{v}(x,0) \in L_2(\mathbf{R}^n)$ is a nonrandom modification of the function $v(x,s,\omega)|_{s=0}$, where $v = L\varphi \in X^1 \cap C_0$ (in other words, $v(x,0) = v(x,0,\omega)$ for a.e. x, ω).

The proof of Lemma 6.1 follows from equalities (2.4), (2.5) and the equality

$$\begin{aligned}
&\mathbf{E}\{(p(x,s,\omega), v(x,s,\omega))_{H^0} \mid \mathcal{F}_s\} \\
&= \mathbf{E}\bigg\{(p(x,T,\omega), v(x,T,\omega))_{H^0} - \int_s^T dt\bigg[(A^*(x,t,\omega)p(x,t,\omega), v(x,t,\omega))_{H^0} \\
&\quad - \bigg(p(x,t,\omega), A(x,t,\omega)v(x,t,\omega) + \sum_{j=1}^d \frac{\partial \chi_j}{\partial x}(x,t,\omega)\beta_j(x,t,\omega) + \varphi(x,t,\omega)\bigg)_{H^0} \\
&\quad - \sum_{j=1}^d ((\nabla, \beta_j(x,t,\omega)p(x,t,\omega)), \chi_j(x,t,\omega))_{H^0}\bigg] \mid \mathcal{F}_s\bigg\}.
\end{aligned}$$

Proof of Theorem 6.1. Let $\xi \in L^2([0,\tau]\times\Omega, \overline{\mathcal{P}}, \lambda\times\mathbf{P}, \mathbf{R})$ be an arbitrary function. Consider the functions $\tilde{\varphi}(x,t,\omega) = \varphi(x,t,\omega)\xi(t,\omega)$ and $v = v(x,s,\omega) = L\tilde{\varphi} \in X^1 \cap C_0$. We have $v(x,0,\omega) \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}, L_2(D))$. The probability of any event of $\mathcal{F}_0$ is equal to 0 or 1. Thus the function $v(x,0,\omega)$ has a nonrandom modification $\bar{v}(x,0) \in L_2(D)$ such that $v(x,0) = v(x,0,\omega)$ for a.e. x, ω.

For $x \in D$ we consider the $(n+1)$-dimensional process

$$\eta^x(t,\omega) = \|y^{x,0}(t,\omega), z^x(t,\omega)\|, \qquad \text{where} \quad z^x(t,\omega) = \int_0^t \tilde{\varphi}[y^{x,0}(\rho,\omega), \rho, \omega]\, d\rho.$$

Analogously we define the process $\eta^{x(\omega)}(t,\omega)$ for a random vector $x(\omega)$ using the process $y^{x(\omega),0}(t,\omega)$ instead of $y^{x,0}(t,\omega)$.

On functions of the form $\eta(t) = \|y(t), z(t)\|$, where $y(\cdot) \in C([0,\tau] \to \mathbf{R}^n)$ and $z(\cdot) \in C([0,\tau] \to \mathbf{R})$, we define the functional $F[\eta(\cdot)] = z(\tau)$, where $\tau = \min\{T, \inf\{t\colon\ y(t) \notin \overline{D}\}\}$. By Theorem 5.1, $\bar{v}(x,0) = \mathbf{E}F[\eta^x(\cdot,\omega)]$ for a.e. x. By virtue of Theorem II.9.4 of [1] establishing an analogue of the Markov property for Itô processes, we have $\mathbf{E}\bar{v}[x(\omega),0] = \mathbf{E}F[\eta^{x(\omega)}(\cdot,\omega)]$. Thus

$$\int_D p_0(x)\bar{v}(x,0)\, dx = \mathbf{E}\bar{v}[x(\omega),0] = \mathbf{E}\int_0^{\tau^{x(\omega),0}(\omega)} \tilde{\varphi}[y^{x(\omega),0}(t,\omega), t, \omega]\xi(t,\omega)\, dt.$$

From these equalities and equality (6.3), where $s = 0$, we obtain

$$\begin{aligned}
&\mathbf{E}\int_0^T \left(\int_D p(x,t,\omega)\varphi(x,t,\omega)\, dx\right)\xi(t,\omega)\, dt \\
&\quad = \mathbf{E}\int_0^T I_\tau(t,\omega)\varphi[y^{x(\omega),0}(t,\omega), t, \omega]\xi(t,\omega)\, dt.
\end{aligned}$$

Since ξ is arbitrary, this relation implies the assertion of the theorem.

Proof of Theorem 6.2. The existence of a nonrandom modification for the function $v(x,0,\omega)$ can be established as in the proof of Theorem 6.1. By virtue of Theorem 5.3.1 of [2] the left-hand member of equality (6.5) coincides with $\mathbf{E}\int_Q p(x,t,\omega) \times\varphi(x,t)\, dx\, dt$. By Lemma 6.1 this value is equal to the middle member of equality (6.5) (and hence to the right-hand member of this equality). The theorem has been proved.

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