Cordes conditions and some alternatives for parabolic equations and discontinuous diffusion*

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Introduction

We consider parabolic equations in nondivergent form with discontinuous coefficients at higher derivatives. Their investigation is most complicated because, in general, in the case of discontinuous coefficients, the uniqueness of a solution for nonlinear parabolic or elliptic equations can fail, and there is no a priory estimate for partial derivatives of a solution. There are some conditions that ensure regularity of solutions of boundary value problems for second order equations and that are known as Cordes conditions (see Cordes (1956)). These conditions restricts the scattering of the eigenvalues of the matrix of the coefficients at higher derivatives. Related conditions from Talenti (1965), Koshelev (1982), Kalita (1989), Landis (1998), on the eigenvalues are also called Cordes type conditions. Gihman and Skorohod (1975) obtained a closed condition implicitly as a part of the proof of the uniqueness of a weak solution in Section 3 of Chapter 3. Cordes (1956) considered elliptic equations. Landis (1998) considered both elliptic and parabolic equations. Koshelev (1982) considered systems of elliptic equations of divirgent type and Hölder property of solutions. Kalita (1989) considered union of divergent and nondivirgent cases.

Conditions from Cordes (1956) are such that they are not necessary satisfied even for constant non-degenerate matrices b, therefore, the condition for $b = b(x)$ means that the corresponding inequalities are satisfied for all x_0 for some non-degenerate matrix $\theta(x_0)$ and

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$\widetilde{b}(x) = \theta(x_0)^T b(x)\theta(x_0)$, where x is from ε-neighborhood of x_0 ($\varepsilon > 0$ is given). We found another condition (Condition 1.1 below) that ensures solvability and uniqueness for first boundary value problem for nondivirgent parabolic equation with discontinuous diffusion coefficients. This condition ensures existence of L_2-integrable derivatives for the solution for L_2-integrable free term. Prior estimate is proved, in contrast with the existing literature.

For discontinuous diffusions, uniqueness of a weak solution cannot be guarantied for the general case (some cases of uniqueness are described in Gihman and Skorohod (1975), Krylov (1980), Anulova *et al* (1998), Liptser and Shiryaev (2000). We obtain some new conditions of uniqueness closed to conditions Gihman and Skorohod (1975) but sometimes less restrictive, as is shown by an example.

Some definitions

Assume that we are given $T > 0$ and an open domain $D \subset \mathbf{R}^n$ such that either $D = \mathbf{R}^n$ or D is bounded with the boundary ∂D that is either C^2-smooth (or such as described in Chapter III.8 in Ladyzhenskaya and Ural'tseva (1968)).

We denote Euclidean norm as $|\cdot|$, and $\bar{D}$ denotes the closure of a region D.

We denote by $\|\cdot\|_X$ the norm in a linear normed space X, and $(\cdot,\cdot)_X$ denotes the scalar product in a Hilbert space X.

Introduce some spaces of functions. Let $G \subset \mathbf{R}^k$ be an open domain, then $W_q^m(G)$ denotes the Sobolev space of functions that belong $L_q(G)$ together with first m derivatives, $q \geq 1$.

Let $H^0 \triangleq L_2(D)$ be the Hilbert space of complex valued functions, and let $H^1 \triangleq \overset{0}{W}{}_2^1(D)$ be the closure in the $W_1^1(D)$-norm of the set of all smooth functions that vanish in a neighborhood of ∂D, $k = 1, 2$. Let $H^2 = W_2^2(D) \cap H^1$ be the space equipped with the norm of $W_2^2(D)$.

Let ℓ_m denotes the Lebesgue measure in $\mathbf{R}^m$, and let $\overline{\mathcal{B}}_m$ be the σ–algebra of the Lebesgue sets in $\mathbf{R}^m$.

We shall use spaces

$X^k = L^2([0,T], \bar{\mathcal{B}}_1, \ell_1, H^k)$, $\mathcal{C}^k = C([0,T]; H^k)$, $k = 0, 1, 2$, $Y^k = X^k \cap \mathcal{C}^{k-1}$, $k = 1, 2$, with the norm $\|v\|_{Y^k} = \|v\|_{X^k} + \|v\|_{\mathcal{C}^{k-1}}$.

1 Solvability of boundary value problem

Consider the domain $D \subset \mathbf{R}^n$ such as described above, $n \geq 1$. Let $Q \triangleq D \times [0, T]$, where $T > 0$ is given.

Let

$$Av = \sum_{i,j=1}^{n} b_{ij}(x,t) \frac{\partial^2 v}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} f_i(x,t) \frac{\partial v}{\partial x_i}(x) - \lambda(x,t) v(x), \tag{1.1}$$

where $(x,t) \in Q$.

We are studying the problem in Q

$$\begin{cases} \frac{\partial v}{\partial t} + Av = -\varphi, \\ v(x,t)|_{x \in \partial D} = 0, \quad v(x,T) = \Phi(x). \end{cases} \tag{1.2}$$

Here $b(x,t) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^{n \times n}$, $f(x,t) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$, and $\lambda(x,t) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{C}$ are measurable bounded functions, b_{ij}, f_i, and x_j are the components of b, f, and x.

If $D = \mathbf{R}^n$, then the boundary condition for ∂D vanish in (1.2).

We assume that $b(x,t)$, $f(x,t)$, $\lambda(x,t)$ vanish for $(x,t) \notin D \times [0,T]$.

Let us state the main conditions imposed on the matrix b.

Condition 1.1 *The matrix $b = b^\top$ is symmetric and has the form $b(x,t) = \bar{b}(x,t) + \widehat{b}(x,t)$, where $\bar{b}(x,t) = \bar{b}(x,t)^\top$ is a continuous bounded matrix such that*

$$\delta \triangleq \inf_{(x,t) \in Q,\ \xi \in \mathbf{R}^n} \frac{\xi^T \bar{b}(x,t) \xi}{|\xi|^2} > 0.$$

The matrix function $\widehat{b}(x,t) \in L^\infty(Q; \mathbf{R}^{n \times n})$ is symmetric and such that there exists a set $\mathcal{N} \subseteq \{1, \ldots, n\}$ such that

$$\widehat{b}_{ij} \equiv \widehat{b}_{ji} \equiv 0 \quad \forall i, j : i \notin \mathcal{N}, j \notin \mathcal{N},$$

and there exists a set $\{\gamma_k\}_{k \in \mathcal{N}}$ such that $\gamma_k \in (0, 2)$ for all k and

$$\widehat{\nu} = \left(\sum_{k \in \mathcal{N}} \frac{1}{2\gamma_k} \right) \operatorname*{ess\,sup}_{x,t} \sum_{k \in \mathcal{N}} \left(\sum_{i \in \mathcal{N}} \widehat{b}_{ik}(x,t)^2 + 4 \sum_{i \notin \mathcal{N}} \widehat{b}_{ik}(x,t)^2 + \frac{\gamma_k}{2 - \gamma_k} \widehat{b}_{kk}(x,t)^2 \right) < \delta^2.$$

Remark 1.1 *If $\operatorname{card} \mathcal{N} < n$, then Condition 1.1 allows bigger than for $\mathcal{N} = \{1, \ldots, n\}$ values $\widehat{b}_{ij}$ for $i \in \mathcal{N}$, $j \in \mathcal{N}$. Different γ_k also make this condition less restrictive: for instance, if $\widehat{b}_{kk} \equiv 0$, then we can allow $\gamma_k = 2 - 0$.*

In particular, the condition for $\widehat{\nu}$ is satisfied if

$$\operatorname*{ess\,sup}_{x,t} \sum_{i,k=1}^{n} \widehat{b}_{ik}(x,t)^2 < \frac{\delta^2}{n}.$$

The next condition is not so principal, since it deals with low order coefficients and the continuous part $\bar{b}$.

Condition 1.2 *There exists a domain* $D_1 \subseteq \mathbf{R}^n$ *and functions* $b^{(\varepsilon)}(x,t) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^{n\times n}$, $f^{(\varepsilon)}(x,t) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$, $\lambda^{(\varepsilon)}(x,t) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{C}$, $\varepsilon > 0$, *such that* $\operatorname{mes} D_1 < +\infty$,

$$\begin{aligned}
\nu_b(\varepsilon) &= \|b^{(\varepsilon)} - \bar{b}\|_{L_\infty(Q)} \to 0 \quad \textit{as} \quad \varepsilon \to 0,\\
\bar{\nu}_b(\varepsilon) &= \operatorname*{ess\,sup}_{(x,t)\in Q} |\frac{\partial b^{(\varepsilon)}}{\partial x}(x,t)| < +\infty \quad \forall \varepsilon > 0,\\
\nu_f(\varepsilon) &= \|f^{(\varepsilon)} - f\|_{L_n(Q_1)} + \operatorname*{ess\,sup}_{(x,t)\in Q\setminus Q_1} |f^{(\varepsilon)}(x,t) - f(x,t)| \to 0 \quad \textit{as} \quad \varepsilon \to 0,\\
\bar{\nu}_f(\varepsilon) &= \operatorname*{ess\,sup}_{(x,t)\in Q} |\frac{\partial f^{(\varepsilon)}}{\partial x}(x,t)| < +\infty \quad \forall \varepsilon > 0,\\
\nu_\lambda(\varepsilon) &= \|\lambda^{(\varepsilon)} - \lambda\|_{L_r(Q_1)} + \operatorname*{ess\,sup}_{(x,t)\in Q\setminus Q_1} |\lambda^{(\varepsilon)}(x,t) - \lambda(x,t)| \to 0 \quad \textit{as} \quad \varepsilon \to 0,\\
\bar{\nu}_\lambda(\varepsilon) &= \operatorname*{ess\,sup}_{(x,t)\in Q} |\frac{\partial \lambda^{(\varepsilon)}}{\partial x}(x,t)| < +\infty \quad \forall \varepsilon > 0
\end{aligned}$$

Here $Q_1 \triangleq D_1 \times (0,T)$, $r \triangleq \max(1, n/2)$.

Remark 1.2 *Condition 1.2 is satisfied if* $f, \bar{b}, \lambda$ *are bounded and* D *is bounded. In that case, we can take* $D_1 = D$ *and the Sobolev averages of the functions* $\bar{b}, f, \lambda$ *as* $b^{(\varepsilon)}, f^{(\varepsilon)}, \lambda^{(\varepsilon)}$ *respectively. Note that Condition 1.2 implies that*

$$\|f\|_{L_n(Q_1)} + \operatorname*{ess\,sup}_{(x,t)\in Q\setminus Q_1} |f(x,t)| < +\infty, \qquad \|\lambda\|_{L_r(Q_1)} + \operatorname*{ess\,sup}_{(x,t)\in Q\setminus Q_1} |\lambda(x,t)| < +\infty.$$

We introduce the set of parameters

$$\begin{aligned}
&\mathcal{P} \triangleq \Big(n,\ D,\ T,\ \delta,\ \mathcal{N},\ \{\gamma_k\}_{k\in\mathcal{N}},\\
&\sup_{x,t}|b(x,t)|,\ \sup_{x,t}|f(x,t)|,\ \sup_{x,t}|\lambda(x,t|,\ \widehat{\nu},\ \nu_b(\cdot),\ \bar{\nu}_b(\cdot),\ \nu_f(\cdot),\ \bar{\nu}_f(\cdot),\ \nu_\lambda(\cdot),\ \bar{\nu}_\lambda(\cdot)\Big).
\end{aligned}$$

We have that $\mathcal{P}$ includes $\nu_b(\cdot)$, hence $\mathcal{P}$ depends on the modulus of continuity of $\bar{b}$.

Theorem 1.1 *Assume that Conditions 1.1–1.2 are satisfied. Then problem (1.2) has the unique solution $v \in \mathcal{Y}^2$ for any $\varphi \in L_2(Q)$, $\Phi \in H^1$, and*

$$\|v\|_{\mathcal{Y}^2} \le c(\|\varphi\|_{L_2(Q)} + \|\Phi\|_{H^1}), \tag{1.3}$$

where $c = c(\mathcal{P})$ is a constant that depends on $\mathcal{P}$.

We shall need some auxiliary spaces to prove the theorem. Let $\widehat{H}^2$ be the set of $v \in W_2^2(D) \cap H^1$ with the special norm

$$\|v\|_{\widehat{H}^2} = \left(\sum_{k\in\mathcal{N}} \left\{ \sum_{i=1}^{n} \left\| \frac{\partial^2 v}{\partial x_k \partial x_i} \right\|_{H^0}^2 - \frac{\gamma_k}{2} \left\| \frac{\partial^2 v}{\partial x_k^2} \right\|_{H^0}^2 \right\} \right)^{1/2} + \alpha_1 \|v\|_{W_2^2(D)}. \tag{1.4}$$

Here $\alpha_1 > 0$ is some constant.

Introduce Banach spaces $\widehat{X}^2 = L^2([0,T], \bar{\mathcal{B}}_1, \ell_1, \widehat{H}^2)$ and $\widehat{Y}^2 = \widehat{X}^2 \cap \mathcal{C}^1$ with the norm

$$\|v\|_{\widehat{Y}^2} = \|v\|_{\widehat{X}^2} + \alpha_2 \|v\|_{\mathcal{C}^1}. \tag{1.5}$$

Here $\alpha_2 > 0$ is a constant.

Remark 1.3 *Since $\gamma_k \in (0,2)$ for all k, (3.1.4) defines a norm, the norm $\widehat{H}^2$ is equivalent to the norm $W_2^2(D)$, and the norm $\widehat{Y}^2$ is equivalent to the norm Y^2.*

Therefore, to prove Theorem 1.1, it suffices to prove the following theorem.

Theorem 1.2 *Assume that Conditions 1.1-1.2 are satisfied. Then problem (1.2) has an unique solution $v \in \widehat{Y}^2$ for any $\varphi \in L_2(Q)$ and $\Phi \in H^1$, and*

$$\|v\|_{\widehat{Y}^2} \le c(\|\varphi\|_{L_2(Q)} + \|\Phi\|_{H^1}), \tag{1.6}$$

where $c > 0$ is a constant that depends only on $\mathcal{P}$ and α_1, α_2.

Remark 1.4 *For $D = \mathbf{R}^n$ a closed to Theorem 1.1 was announced in Dokuchaev (1996), where, however, the estimate was obtained for the derivatives with discontinuous coefficients only, just to make the equation meaningful).*

2 Examples

Let $b = b(x)$, and let $\lambda_1,\dots,\lambda_n$ be its eigenvalues. The classic Cordes conditions from Cordes (1956) was formulated for $n \geq 3$ as

$$\exists \varepsilon > 0: \quad (n-1)\sum_{i<j}(\lambda_i - \lambda_j)^2 < (1-\varepsilon)\left(\sum_{i=1}^{n}\lambda_i\right)^2. \tag{2.1}$$

It was shown by Talenti (1965) that (2.1) is equivalent to

$$\exists \varepsilon > 0: \quad (n-1+\varepsilon)\sum_{i=1}^{n}\lambda_i^2 = (n-1+\varepsilon)\sum_{i,j=1}^{n} b_{ij}^2 < \left(\sum_{i=1}^{n} b_{ii}\right)^2 = \left(\sum_{i=1}^{n}\lambda_i\right)^2. \tag{2.2}$$

This form (2.2) can be given also to the condition from Kalita (1989) for a system with one nondivirgent equation.

Conditions from Landis (1998) has the form

$$\exists \varepsilon > 0: \quad \sum_{i=1}^{n}\lambda_i < (n+2-\varepsilon)\min\{\lambda_1,\dots,\lambda_n\}. \tag{2.3}$$

The condition from Section 3, Chapter 3 from Gihman and Skorohod (1975) is such that in the simplest case can be written as

$$\exists \varepsilon > 0: \quad \mathrm{Tr}\left((b-I)^2\right) < 1-\varepsilon. \tag{2.4}$$

(In Gihman and Skorohod (1975), I was replaced for a smooth matrix function).

In our notations, the last condition can be rewritten as

$$\bar{b} \equiv I, \quad \exists \varepsilon > 0: \quad \sum_{i,j=1}^{n}\widehat{b}_{ij}^2 < 1-\varepsilon. \tag{2.5}$$

The regularity of the parabolic equation established by Gihman and Skorohod (1975) under condition (2.4) is weaker than the regularity established by Theorem 1.1

Note that Gihman and Skorohod (1975) obtained the regularity that was just enough to ensure the uniqueness of a weak solution of some Ito's equation. In fact, conditions (2.4), (2.5) are sufficient for Theorem 1.1 as well. We leave it without proof; note that there is a proof similar to the proof given below and different from the one given in Gihman and Skorohod (1975).

In fact, Cordes conditions mean that inequalities (2.1)–(2.3) are satisfied for all x_0 for some non-degenerate matrix $\theta(x_0)$ and for all matrices $\widehat{b}(x) = \theta(x_0)^T b(x)\theta(x_0)$, where x is from the ε-neighborhood of x_0, and where $\varepsilon > 0$ is given. Similarly, condition (2.3) was adjusted in Landis (1998), and condition (2.4) was adjusted in Gihman and Skorohod (1975).

Let $n = 3$, $b(x,t) \equiv b(x)$,

$$b(x) = \begin{pmatrix} 1 & \alpha(x) & \beta(x) \\ \alpha(x) & 1 & 0 \\ \beta(x) & 0 & 1 \end{pmatrix}, \quad \widehat{b}(x) = \begin{pmatrix} 0 & \alpha(x) & \beta(x) \\ \alpha(x) & 0 & 0 \\ \beta(x) & 0 & 0 \end{pmatrix},$$

where $\alpha(x)$,$\beta(x)$ are arbitrary measurable functions, $|\alpha(x)| \le \alpha = \text{const}$, $|\beta(x)| \le \beta = \text{const}$, and functions $\alpha(x)$,$\beta(x)$ are quite irregular.

It is easy to see that Condition 1.1 is satisfied if $\alpha^2 + \beta^2 < 1$ for $\mathcal{N} = \{1\}$ and for some $\gamma_1 < 2$ being close enough to 2.

The spectrum of b is $\{1,\, 1 - \sqrt{\alpha(x)^2 + \beta(x)^2},\, 1 + \sqrt{\alpha(x)^2 + \beta(x)^2}\}$. Then conditions (2.1), (2.2) fails if $(\bar{\alpha}^2 + \bar{\beta}^2) \ge 3/4$, and (2.3) fails if $(\bar{\alpha}^2 + \bar{\beta}^2) \ge 2/5$. Conditions (2.4) and (2.5) fail if $\bar{\alpha}^2 + \bar{\beta}^2 > 1/2$.

Therefore, Condition 1.1 is less restrictive for this example than condition (2.5) or the conditions from Cordes (1956), Gihman and Skorohod (1975), Kalita (1989), Koshelev (1982), Landis (1998), Talenti (1965).

There may be opposite examples when condition (2.1) is satisfied, but Condition 1.1 fails.

3 Proof of Theorem 3.1.2.

The main idea is to prove theorem for some $\varepsilon = \varepsilon(\mathcal{P}) > 0$ for u replaced with

$$u_\varepsilon(x,t) \triangleq u(x,t)\exp\{K(\varepsilon)t\}, \tag{3.1}$$

where $K(\varepsilon) > 0$ is a function of ε such that $K(\varepsilon) \to +\infty$ as $\varepsilon \to +0$.

Let $\widetilde{\lambda}^{(\varepsilon)}(x,t) \triangleq \lambda^{(\varepsilon)}(x,t) + K(\varepsilon)$, and let

$$A_\varepsilon u \triangleq \sum_{i,j=1}^n b_{ij}^{(\varepsilon)}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_i^n f_i^{(\varepsilon)}(x,t)\frac{\partial u}{\partial x_i}(x) - \widetilde{\lambda}^{(\varepsilon)}(x,t)u(x,t).$$

Consider the problem

$$\begin{cases} \frac{\partial v}{\partial t} + A_\varepsilon v = -\varphi, \\ v(x,t)|_{x\in\partial D} = 0, \quad v(x,T) = \Phi(x). \end{cases} \tag{3.2}$$

Introduce the operators $L(\varepsilon): X^0 \to \widehat{Y}^2$, $\mathcal{L}(\varepsilon): H^1 \to \widehat{Y}^2$ such that $u \triangleq L(\varepsilon)\varphi + \mathcal{L}(\varepsilon)\Phi$ is the solution of (3.2). Let $\|L(\varepsilon)\|$ denotes the norm of the operator $L(\varepsilon): X^0 \to \widehat{Y}^2$, and let $\|\mathcal{L}(\varepsilon)\|$ denotes the norm of the operator $\mathcal{L}(\varepsilon): H^1 \to \widehat{Y}^2$.

Lemma 3.1 *For any $\gamma > 0$, there exists a small enough $\varepsilon_* > 0$, and a function $K(\varepsilon) > 0$ (increasing as $\varepsilon \to 0$), and $\alpha_1 > 0$ and $\alpha_2 > 0$ in (1.4)-(1.5), such that $\varepsilon_* = \varepsilon_*(\gamma, \mathcal{P})$, $K(\cdot) = K(\cdot, \gamma, \mathcal{P})$, $\alpha_i = \alpha_i(\gamma, \mathcal{P}, \varepsilon)$, and*

$$\|L(\varepsilon)\| \le \gamma + \frac{1}{\delta}\biggl(\sum_{k\in\mathcal{N}} \frac{1}{2\gamma_k}\biggr)^{1/2}, \quad \|\mathcal{L}(\varepsilon)\| \le c_0 \quad \forall \varepsilon \in (0, \varepsilon_*], \tag{3.3}$$

where $c_0 = c_0(\mathcal{P}, \alpha_1, \alpha_2)$ is a constant.

Proof. Let $\varphi \in X^0$ be a smooth function with a compact support inside Q.

Let $v = L(\varepsilon)\varphi$. We have

$$\frac{1}{2}\|v(\cdot, t_1)\|^2_{H^0} = \frac{1}{2}\|v(\cdot, T)\|^2_{H^0} + \int_{t_1}^T (v, A_\varepsilon v + \varphi)_{H^0} ds. \tag{3.4}$$

We shall use below the obvious inequality

$$2\alpha\beta \le \varepsilon\alpha^2 + \varepsilon^{-1}\beta^2 \quad \forall \alpha, \beta, \varepsilon \in \mathbf{R}, \varepsilon > 0.$$

In particular,

$$(v, \varphi)_{H^0} \le \frac{1}{2\varepsilon_1}\|v\|^2_{H^0} + \frac{\varepsilon_1}{2}\|\varphi\|^2_{H^0} \quad \forall \varepsilon_1 > 0. \tag{3.5}$$

We have the estimate

$$\begin{aligned}(v, A_\varepsilon v + \varphi)_{H^0} &= \Bigl(v, \sum_{i,j=1}^n b_{ij}^{(\varepsilon)}(\cdot, t)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^n f_i^{(\varepsilon)}(\cdot, t)\frac{\partial v}{\partial x_i} - \widetilde{\lambda}^{(\varepsilon)}(\cdot, t) v(\cdot, t)\Bigr)_{H^0} \\ &= \sum_{i,j=1}^n \Bigl\{ -\Bigl(v, \frac{\partial b_{ij}^{(\varepsilon)}}{\partial x_j}\frac{\partial v}{\partial x_i}\Bigr)_{H^0} - \Bigl(\frac{\partial v}{\partial x_j}, b_{ij}^{(\varepsilon)}\frac{\partial v}{\partial x_i}\Bigr)_{H^0}\Bigr\} \\ &\quad -\frac{1}{2}\Bigl(v^2, \sum_{i=1}^n \frac{\partial f_i^{(\varepsilon)}}{\partial x_i}\Bigr)_{H^0} - (v, \lambda^{(\varepsilon)} v)_{H^0} - K(\varepsilon)\|v\|^2_{H^0} + (v, \varphi)_{H^0} \\ &\le (-\delta + \nu_1)\sum_{j=1}^n \Bigl\|\frac{\partial v}{\partial x_j}\Bigr\|^2_{H^0} - K(\varepsilon)\|v\|^2_{H^0} + c_1\|v\|^2_{H^0} + \frac{\varepsilon_1}{2}\|\varphi\|^2_{H^0},\end{aligned}$$

where $\varepsilon_1 > 0$, $\nu_1 > 0$ can be arbitrarily small, and c_1 depends on ε, ε_1, ν_1, $\mathcal{P}$. Hence we have that choosing $K(\varepsilon) = K(\varepsilon, \nu) > c_1$ for $\nu > 0$ can ensure that

$$\|L(\varepsilon)\varphi\|_{\widehat{Y}^1} \le \nu\|\varphi\|_{X^0} \quad \forall \varepsilon \in (0, \varepsilon_*], \quad \forall \varphi \in X^0. \tag{3.6}$$

We have that

$$\left\|\frac{\partial v}{\partial x_k}(\cdot,t_1)\right\|_{H^0}^2-\left\|\frac{\partial v}{\partial x_k}(\cdot,T)\right\|_{H^0}^2=2\int_{t_1}^T\left(\frac{\partial v}{\partial x_k},\frac{\partial}{\partial x_k}(A_\varepsilon v+\varphi)\right)_{H^0}ds. \tag{3.7}$$

Remind that φ has compact support inside Q. Then

$$\left(\frac{\partial v}{\partial x_k},\frac{\partial \varphi}{\partial x_k}\right)_{H^0}\le\frac{\delta\gamma_k}{2}\left\|\frac{\partial^2 v}{\partial x_k^2}\right\|_{H^0}^2+\frac{1}{2\delta\gamma_k}\|\varphi\|_{H^0}^2. \tag{3.8}$$

Note that if $b^{(\varepsilon)}\in C^2$ then

$$\begin{aligned}&\left(\tfrac{\partial v}{\partial x_k},\tfrac{\partial b_{ij}^{(\varepsilon)}}{\partial x_k}\tfrac{\partial^2 v}{\partial x_i\partial x_j}\right)_{H^0}\\ &=-\left(\tfrac{\partial v}{\partial x_k},\tfrac{\partial^2 b_{ij}^{(\varepsilon)}}{\partial x_i\partial x_k}\tfrac{\partial v}{\partial x_j}\right)_{H^0}-\left(\tfrac{\partial^2 v}{\partial x_i\partial x_k},\tfrac{\partial b_{ij}^{(\varepsilon)}}{\partial x_k}\tfrac{\partial v}{\partial x_j}\right)_{H^0}+\textstyle\int_{\partial D}\widehat{J}_{ijk}ds\\ &=\left(\tfrac{\partial v}{\partial x_k},\tfrac{\partial b_{ij}^{(\varepsilon)}}{\partial x_i}\tfrac{\partial^2 v}{\partial x_j\partial x_k}\right)_{H^0}+\left(\tfrac{\partial^2 v}{\partial x_k^2},\tfrac{\partial b_{ij}^{(\varepsilon)}}{\partial x_i}\tfrac{\partial v}{\partial x_j}\right)_{H^0}-\left(\tfrac{\partial^2 v}{\partial x_i\partial x_k},\tfrac{\partial b_{ij}^{(\varepsilon)}}{\partial x_k}\tfrac{\partial v}{\partial x_k}\right)_{H^0}+\textstyle\int_{\partial D}J'_{ijk}ds,\end{aligned} \tag{3.9}$$

where

$$J'_{ijk}=\widehat{J}_{ijk}-\frac{\partial\bar v}{\partial x_k}\frac{\partial b_{ij}^{(\varepsilon)}}{\partial x_i}\frac{\partial v}{\partial x_j}\cos(\mathbf{n},e_k),\quad \widehat{J}_{ijk}=\frac{\partial\bar v}{\partial x_k}\frac{\partial b_{ij}^{(\varepsilon)}}{\partial x_k}\frac{\partial v}{\partial x_j}\cos(\mathbf{n},e_i),$$

$\mathbf{n}=\mathbf{n}(s)$ is the outward pointing normal to the surface ∂D at the point $s\in\partial D$, and e_k is the kth basis vector in the Euclidean space $\mathbf{R}^n=\{x_1,\dots,x_n\}$.

If $b^{(\varepsilon)}$ is general, then the right hand and the left hand expressions in (3.9) are still equal. Hence, we obtain

$$\begin{aligned}&\left(\tfrac{\partial v}{\partial x_k},\tfrac{\partial b_{ij}^{(\varepsilon)}}{\partial x_k}\tfrac{\partial^2 v}{\partial x_i\partial x_j}\right)_{H^0}\\ &\le\varepsilon_2\left(\left\|\tfrac{\partial^2 v}{\partial x_k\partial x_j}\right\|_{H^0}^2+\left\|\tfrac{\partial^2 v}{\partial x_k\partial x_i}\right\|_{H^0}^2+\left\|\tfrac{\partial^2 v}{\partial x_k^2}\right\|_{H^0}^2\right)+c_2\tfrac{1}{2\varepsilon_2}\|v\|_{H^1}^2+\textstyle\int_{\partial D}J'_{ijk}ds\quad\forall\varepsilon_2>0,\end{aligned} \tag{3.10}$$

where the constant c_2 depends only on $\mathcal{P}$.

Therefore,

$$\begin{aligned}&\left(\frac{\partial v}{\partial x_k},\frac{\partial}{\partial x_k}(A_\varepsilon v+\varphi)\right)_{H^0}\\ &=\left(\frac{\partial v}{\partial x_k},\frac{\partial}{\partial x_k}\left\{\sum_{i,j=1}^n b_{ij}^{(\varepsilon)}(\cdot,t)\frac{\partial^2 v}{\partial x_i\partial x_j}+\sum_{i=1}^n f_i^{(\varepsilon)}(\cdot,t)\frac{\partial v}{\partial x_i}-\widetilde{\lambda}^{(\varepsilon)}(\cdot,t)v(\cdot,t)+\varphi(\cdot,t)\right\}\right)_{H^0}\\ &\qquad=\sum_{i,j=1}^n\left\{\left(\frac{\partial v}{\partial x_k},\frac{\partial b_{ij}^{(\varepsilon)}}{\partial x_k}\frac{\partial^2 v}{\partial x_i\partial x_j}\right)_{H^0}-\left(\frac{\partial^2 v}{\partial x_k\partial x_i},b_{ij}^{(\varepsilon)}\frac{\partial^2 v}{\partial x_k\partial x_j}\right)_{H^0}\right\}\\ &\qquad\quad+\sum_{i=1}^n\left\{\left(\frac{\partial v}{\partial x_k},\frac{\partial f^{(\varepsilon)}}{\partial x_k}\frac{\partial v}{\partial x_i}\right)_{H^0}+\left(\frac{\partial v}{\partial x_k},f_i^{(\varepsilon)}\frac{\partial^2 v}{\partial x_k\partial x_i}\right)_{H^0}\right\}\end{aligned}$$

$$-\left(\frac{\partial v}{\partial x_k},\frac{\partial\lambda^{(\varepsilon)}}{\partial x_k}v+\lambda^{(\varepsilon)}\frac{\partial v}{\partial x_k}\right)_{H^0}-K(\varepsilon)\left\|\frac{\partial v}{\partial x_k}\right\|_{H^0}^2+\left(\frac{\partial v}{\partial x_k},\frac{\partial\varphi}{\partial x_k}\right)_{H^0}+\int_{\partial D}J_{ijk}ds$$
$$\leq(-\delta+\nu_2+2\varepsilon_3)\sum_{j=1}^n\left\|\frac{\partial^2 v}{\partial x_k\partial x_j}\right\|_{H^0}^2+\left(\frac{\delta\gamma_k}{2}+\varepsilon_3\right)\left\|\frac{\partial^2 v}{\partial x_k^2}\right\|_{H^0}^2+c_2\|v\|_{H^1}^2$$
$$+\sum_{i,j=1}^n\int_{\partial D}J_{ijk}ds+\left(\frac{1}{2\delta_k}+\frac{\varepsilon_1}{2}\right)\|\varphi\|_{H^0}^2-K(\varepsilon)\left\|\frac{\partial v}{\partial x_k}\right\|_{H^0}^2,\quad (3.11)$$

where the constant c_2 depends only on $\mathcal{P}$, constants $\varepsilon_3>0$ and $\nu_2>0$ can be arbitrarily small,

$$J_{ijk}=J'_{ijk}+J''_{ijk},\quad J''_{ijk}=\frac{\partial\bar v}{\partial x_k}b_{ij}^{(\varepsilon)}\frac{\partial^2 v}{\partial x_i\partial x_j}\cos(\mathbf{n},e_k).$$

Let us estimate $\int_{\partial D}J_{ijk}$. It vanishes if $D=\mathbf{R}^n$ (as well as all integrals over the boundary ∂D). For a bounded domain D, we mainly follow the approach from Section 3.8 Ladyzhenskaya and Ural'tseva (1968). Let $x^0=\{x_i^0\}_{i=1}^n\in\partial D$ be an arbitrary point. In its neighborhood, we introduce local Cartesian coordinates $y_m=\sum_{k=1}^n c_{mk}(x_k-x_k^0)$ such that the axis y_n is directed along the outward normal $\mathbf{n}=\mathbf{n}(x_0)$ and $\{c_{mk}\}$ is an orthogonal matrix.

Let $y_n=\omega(y_1,\ldots,y_{n-1})$ be an equation determining the surface ∂D in a neighborhood of the origin. By the properties of the surface ∂D, the first order and second order derivatives of the function ω are bounded. Since $\{c_{mk}\}$ is an orthogonal matrix, we have $x_k-x_k^0=\sum_{m=1}^n c_{km}y_m$. Therefore, $\cos(\mathbf{n},e_m)=c_{nm}$, $m=1,\ldots,n$. Then

$$J'_{ijk}=\sum_{m=1}^n c_{mk}\frac{\partial\bar v}{\partial y_m}\sum_{p=1}^n c_{pi}\frac{\partial v}{\partial y_p}\left(\sum_{q=1}^n\frac{\partial b_{ij}^{(\varepsilon)}}{\partial y_q}c_{qk}c_{ni}-\sum_{r=1}^n\frac{\partial b_{ij}^{(\varepsilon)}}{\partial y_r}c_{rk}c_{nk}\right),$$
$$J''_{ijk}=\sum_{m=1}^n c_{mk}\frac{\partial\bar v}{\partial y_m}b_{ij}c_{nk}\sum_{p,q=1}^n c_{pi}c_{qi}\frac{\partial^2 v}{\partial y_p\partial y_q}.\quad (3.12)$$

The boundary condition $v(x,t)|_{x\in\partial D}=0$ has the form

$$v(y_1,\ldots,y_{n-1},\omega(y_1,\ldots,y^{n-1}),t)=0$$

identically with respect to $y_1,\ldots,y_{n-1}$ near the point $y_1=\ldots=y_{n-1}=0$. Let us differentiate this identity with respect to y_p and y_q, $p,q=1,\ldots,n-1$, and take into account that

$$\frac{\partial\omega}{\partial y_p}=0\quad(p=1,\ldots,n-1).$$

at x_0. Then

$$\frac{\partial v}{\partial y_p}=0,\quad\frac{\partial^2 v}{\partial y_p\partial y_q}=-\frac{\partial v}{\partial y_n}\frac{\partial^2\omega}{\partial y_p\partial y_q}=-\frac{\partial v}{\partial\mathbf{n}}\frac{\partial^2\omega}{\partial y_p\partial y_q}\quad(p,q=1,\ldots,n-1).$$

Hence

$$\begin{aligned}\textstyle\int_{\partial D} J_{ijk} ds &\le \widehat{c}_1 \textstyle\int_{\partial D} \left|\frac{\partial v}{\partial \mathbf{n}}\right|^2 ds \\ &\le \varepsilon_4 \textstyle\sum_{i,j=1}^n \int_D \left|\frac{\partial^2 v}{\partial x_i \partial x_j}(x)\right|^2 dx + \widehat{c}_2(1+\varepsilon_4^{-1})\|v\|_{H^1}^2 \quad \forall \varepsilon_4 > 0\end{aligned} \tag{3.13}$$

for some constants $\widehat{c}_i = \widehat{c}_i(\varepsilon, \mathcal{P})$. The last estimate follows from the estimate (2.38) in Chapter 2 from Ladyzhenskaya and Ural'tseva (1968).

As mentioned above, for a suitable choice of the functions $K(\varepsilon) = K(\varepsilon, \nu)$ and for an arbitrarily small $\nu > 0$, one can provide the estimate $\|L(\varepsilon)\varphi\|_{\mathcal{Y}^1} \le \nu\|\varphi\|_{X^0}$ $(\forall \varepsilon \in (0, \varepsilon_*], \forall \varphi \in X^0)$. The constants $\varepsilon_3 > 0$, $\varepsilon_4 > 0$, and $\nu_2 > 0$ can be arbitrarily small, and the constant c_1 depends on $\varepsilon, \varepsilon_1, \nu_1, \gamma_k$ and , $\mathcal{P}$. Combining (3.6) with (3.11) and (3.13), we see that for some function $K(\varepsilon)$ we have

$$\begin{aligned}\sum_{k\in\mathcal{N}} \Bigg(\int_0^T dt \int_D \Bigg\{ (\delta - \nu_2 - 2\varepsilon_3) \sum_{i=1}^n \left|\frac{\partial^2 v}{\partial x_k \partial x_i}(x,t)\right|^2 - \left(\frac{\delta\gamma_k}{2} + \varepsilon_3\right) \left|\frac{\partial^2 v}{\partial x_k^2}(x,t)\right|^2 \Bigg\} dx \\ + \frac{1}{2}\sup_t \left\|\frac{\partial v}{\partial x_k}(\cdot,t)\right\|_{H^0}^2 \Bigg) \\ \le \sum_{k\in\mathcal{N}} \left(c_2\|v\|_{X^1}^2 + \left(\frac{1}{2\delta\gamma_k} + \frac{\varepsilon_1}{2}\right)\|\varphi\|_{X^0}^2\right) \le \sum_{k\in\mathcal{N}} \left(\nu c_2 + \frac{1}{2\delta\gamma_k} + \frac{\varepsilon_1}{2}\right)\|\varphi\|_{X^0}^2.\end{aligned} \tag{3.14}$$

Therefore,

$$\begin{aligned}\sum_{k\in\mathcal{N}} \Bigg(\int_0^T \int_D dt \Bigg\{ (\delta - \varepsilon_5) \sum_{i=1}^n \left|\frac{\partial^2 v}{\partial x_k \partial x_i}(x,t)\right|^2 - \frac{\delta\gamma_k}{2} \left|\frac{\partial^2 v}{\partial x_k^2}(x,t)\right|^2 \Bigg\} dx \\ + \frac{1}{2}\sup_t \left\|\frac{\partial v}{\partial x_k}(\cdot,t)\right\|_{H^0}^2 \Bigg) \le \sum_{k\in\mathcal{N}} \left(\frac{1}{2\delta\gamma_k} + \varepsilon_6\right)\|\varphi\|_{X^0}^2\end{aligned}$$

for some sufficiently small $\varepsilon_i = \varepsilon_i(\varepsilon, \mathcal{P}) > 0$, $i = 5, 6$. (Here ν_2, ε_3 are from (3.11)). Take the sum in (3.14) with respect to $k = 1, \ldots, n$ and choose a sufficiently small number $\alpha_1 = \alpha_1(\varepsilon)$. This, together with (3.14), yields the first estimate in (3.3).

In a similar way, taking into account the initial condition in (3.14) and taking the sum in (3.14) with respect to $k = 1, \ldots, n$, we obtain the estimate

$$\|v\|_{\widehat{Y}^2} \le \widetilde{c}\|\Phi\|_{H^1}$$

for $v = \mathcal{L}(\varepsilon)\Phi$, where $\widetilde{c} = \widetilde{c}(\mathcal{P})$ is a constant. Then we obtain the assertion of Lemma 3.1. $\square$

Introduce the operator $R(\varepsilon) : \widehat{Y}^2 \to \widehat{Y}^2$

$$R(\varepsilon)v = L(\varepsilon)\Big\{\sum_{i,j=1}^{n} \widehat{b}_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n}[\bar{b}_{ij} - b_{ij}^{(\varepsilon)}]\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n}[f_i - f_i^{(\varepsilon)}]\frac{\partial v}{\partial x_i} - [\lambda - \lambda^{(\varepsilon)}]v\Big\}. \quad (3.15)$$

Lemma 3.2 *There exists a number $\bar{\varepsilon} = \bar{\varepsilon}(\mathcal{P}) > 0$ such that the norm of the operator $R(\varepsilon) : \widehat{Y}^2 \to \widehat{Y}^2$ can be estimated as $\|R(\varepsilon)\| < 1$ $(\forall \varepsilon \in (0, \bar{\varepsilon}])$.*

Proof. We have

$$\begin{aligned}
\Big|\sum_{i,j=1}^{n} \widehat{b}_{ij}(x,t)\frac{\partial^2 v}{\partial x_i \partial x_j}(x,t)\Big|^2 &= \Big|\sum_{k\in\mathcal{N}}\Big(\sum_{i\in\mathcal{N}} \widehat{b}_{ki}(x,t)\frac{\partial^2 v}{\partial x_k \partial x_i}(x,t) + 2\sum_{i\notin\mathcal{N}} \widehat{b}_{ki}(x,t)\frac{\partial^2 v}{\partial x_k \partial x_i}(x,t)\Big)\Big|^2 \\
&\le \Big|\sum_{k\in\mathcal{N}}\Big\{\Big(\sum_{i\in\mathcal{N}, i\neq k} \widehat{b}_{ki}(x,t)^2 + 4\sum_{i\notin\mathcal{N}} \widehat{b}_{ki}(x,t)^2 + \Big[1 - \frac{\gamma_k}{2}\Big]^{-1} \widehat{b}_{kk}(x,t)^2\Big)^{1/2} \\
&\qquad \times \Big(\sum_{i=1,\dots,n,\ i\neq k}\Big|\frac{\partial^2 v}{\partial x_k \partial x_i}(x,t)\Big|^2 + \Big[1 - \frac{\gamma_k}{2}\Big]\Big|\frac{\partial^2 v}{\partial x_k^2}(x,t)\Big|^2\Big)^{1/2}\Big\}\Big|^2 \\
&\le \sum_{k\in\mathcal{N}}\Big(\sum_{k\in\mathcal{N}, i\neq k} \widehat{b}_{ki}(x,t)^2 + 4\sum_{i\notin\mathcal{N}} \widehat{b}_{ki}(x,t)^2 + \Big[1 - \frac{\gamma_k}{2}\Big]^{-1} \widehat{b}_{kk}(x,t)^2\Big) \\
&\qquad \times \sum_{k\in\mathcal{N}}\Big(\sum_{i=1,\dots,n,\, i\neq k}\Big|\frac{\partial^2 v}{\partial x_k \partial x_i}(x,t)\Big|^2 + \Big[1 - \frac{\gamma_k}{2}\Big]\Big|\frac{\partial^2 v}{\partial x_k^2}(x,t)\Big|^2\Big). \quad (3.16)
\end{aligned}$$

Hence

$$\Big\|\sum_{i,j=1}^{n} \widehat{b}_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j}\Big\|_{X^0}^2 \le \widehat{\nu}\Big(\sum_{k\in\mathcal{N}}\frac{1}{2\gamma_k}\Big)^{-1}\|v\|_{\widehat{X}^2}^2 < \delta^2\Big(\sum_{k\in\mathcal{N}}\frac{1}{2\gamma_k}\Big)^{-1}\|v\|_{\widehat{X}^2}^2.$$

In addition, Condition 1.2 and the embedding theorems for Sobolev spaces imply the estimates

$$\begin{aligned}
\Big\|\sum_{i,j=1}^{n}(\bar{b}_{ij} - b_{ij}^{(\varepsilon)})\frac{\partial^2 v}{\partial x_i \partial x_j}\Big\|_{X^0} + \Big\|\sum_{i=1}^{n}(f_i - f_i^{(\varepsilon)})\frac{\partial v}{\partial x_i}\Big\|_{X^0} + \Big\|(\lambda - \lambda^{(\varepsilon)})v\Big\|_{X^0} \\
\le C(\nu_b(\varepsilon) + \nu_f(\varepsilon) + \nu_\lambda(\varepsilon))\|v\|_{\widehat{X}^2},
\end{aligned}$$

where the constant C depends only on n. This proves Lemma 3.2. □

Let us now complete the proof of Theorem 1.2. By Lemma 3.2, $(I - R(\varepsilon))^{-1} : \widehat{Y}^2 \to \widehat{Y}^2$ is a continuous operator. Let

$$\varphi_\varepsilon(x,t) \triangleq \varphi(x,t)e^{K(\varepsilon)t}. \quad (3.17)$$

The function $u(x,t)$ is the desired solution of problem (1.2), if relation (3.1) holds, where

$$u_\varepsilon = (I - R(\varepsilon))^{-1}[L(\varepsilon)\varphi_\varepsilon + \mathcal{L}(\varepsilon)\Phi] \quad (3.18)$$

because we have

$$u_\varepsilon \triangleq L(\varepsilon)\varphi_\varepsilon + \mathcal{L}(\varepsilon)\Phi + R(\varepsilon)u_\varepsilon.$$

in view of (3.17)–(3.18). Therefore,

$$\|u_\varepsilon\|_{\widehat{Y}^2} \le (1 - \|R(\varepsilon)\|)^{-1}(\|L(\varepsilon)\|\,\|\varphi_\varepsilon\|_{L_2(Q)} + \|\mathcal{L}(\varepsilon)\|\|\Phi\|_{H^1}).$$

This, together with (3.1) yields the estimate (1.6) and the assertion of Theorem 1.2. □

4 Uniqueness of a weak solution of Itô's equation

Consider the n-dimensional vector Itôs equation

$$\begin{cases} dy(t) = f(y(t),t)dt + \beta(y(t),t)dw(t), \\ y(s) = a. \end{cases} \tag{4.1}$$

By $y^{a,s}(t)$ we denote a solution of this equation, $0 \le s \le t \le T$.

In (4.1), $w(t)$ is a Wiener process of dimension n, $f(x,t) : Q \to \mathbf{R}^n$, $\beta(x,t) : Q \to \mathbf{R}^{n\times n}$, $Q = \mathbf{R}^n \times (0,T)$ are measurable functions.

Denote

$$b(x,t) \triangleq \frac{1}{2}\beta(x,t)\beta(x,t)^{\mathrm{T}}.$$

We assume that the functions $f(x,t)$, $\beta(x,t)$, $b(x,t)$ are bounded and that the function b satisfies Condition 1.1.

Let $(\Omega_0, \mathcal{F}_0, \mathbf{P}_0)$ be a probability space.

Theorem 4.1 *(Krylov (1980), Chapter 2). For any random variable $a \in L^2(\Omega_0, \mathcal{F}_0, \mathbf{P}_0, \mathbf{R}^n)$, there exists a set*

$$\Big\{(\Omega, \mathcal{F}, \mathbf{P}), (w(t), \mathcal{F}_t), y^{a,s}(t)\Big\},$$

where $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space such that $a \in L^2(\Omega, \mathcal{F}, \mathbf{P})$, $(w(t), \mathcal{F}_t)$ is a Wiener process of dimension n on $(\Omega, \mathcal{F}, \mathbf{P})$, $\mathcal{F}_t \subseteq \mathcal{F}$ is a filtration of σ-algebras of events such that $w(t) - w(s)$ do not depend on a and on $\mathcal{F}_s$ for $t > s$, and $y^{a,s}(t)$ is the solution of (4.1) for $w(t)$.

(In the cited book, the proof was given for non-random a, which is unessential).

We assume that $Q = D \times (0,T)$, where either or $D = \mathbf{R}^n$ or $D \subseteq \mathbf{R}^n$ is a bounded simply connected domain with C^2-smooth boundary.

Introduce a bounded measurable function $\lambda(x,t) : Q \to \mathbf{C}$. We assume the following condition.

Condition 4.1 *The functions b, f, λ are such that the conclusion of Theorem 1.1 is valid.*

Remark 4.1 *It follows from Theorem 1.1 that Condition 4.1 is satisfied if Condition 1.1 is satisfied for b, and Condition 1.2 is satisfied for f and λ.*

Let χ denotes the indicator function.

Theorem 4.2 *Let a be a random vector with the probability density function $\rho(x)$, let $a \in D$ a.s., $\rho \in H^{-1}$, and $\mathbf{E}|a|^2 < +\infty$. Let functions $f(x,t)$, $\beta(x,t)$, $b(x,t)$ be measurable and bounded, and let Condition 4.1 be satisfied. Let $y^{a,s}(t)$ be a weak solution of (4.1), $\tau^{a,s} \triangleq \inf\{t : y^{a,s}(t) \notin D\}$. For the functions $\varphi \in L_2(Q)$ and $\Phi \in H^1$, set*

$$\begin{aligned} F_{a,s} \triangleq \quad & \mathbf{E}\Phi(y^{a,s}(T)) \exp\Big\{-\textstyle\int_s^{\tau^{a,s}\wedge T} \lambda(y^{a,s}(r),r)dr\Big\}\chi_{\{\tau^{a,s}\geq T\}} \\ & + \mathbf{E}\textstyle\int_s^{\tau^{a,s}\wedge T} \varphi(y^{a,s}(t),t) \exp\Big\{-\int_s^t \lambda(y^{a,s}(r),r)dr\Big\}dt. \end{aligned} \tag{4.2}$$

Then

$$F_{a,s} = (v(\cdot,s),\rho)_{H^0}, \tag{4.3}$$

where $v \in Y^2$ is a (unique) solution of problem (1.2) for the operator A given by formula (1.1) with the above functions f, b and λ, and

$$|F_{a,s}| \leq c\|\rho\|_{H^{-1}}(\|\varphi\|_{L_2(Q)} + \|\Phi\|_{H^1}),$$

where $c = c(\mathcal{P})$ is a constant occurring in Theorem 1.2.

Corollary 4.1 *(The Maximum Principle). Assume that conditions of Theorem 4.2 are satisfied and, in addition, that λ is a real function, $\varphi(x,t) \geq 0$ for a.e. x, t, and $\Phi(x) \geq 0$ for a.e. x. Then the solution v of problem (1.2) is such that $v(x,t) \geq 0$ for all t for a.e. x.*

Introduce operators $L_{s,t} : L_2(D \times (s,t)) \to H^1$, $\mathcal{L}_{s,t} : H^1 \to H^1$ such that $v(\cdot,s) = L_{s,t}\varphi + \mathcal{L}_{s,t}\Phi$ is the solution of the problem

$$\begin{cases} \frac{\partial v}{\partial r}(x,r) + Av(x,r) = -\varphi(x,r), \quad r < t, \\ v(x,r)|_{x\in\partial D} = 0, \quad v(x,t) = \Phi(x). \end{cases}$$

at the instant s, where $s < t$. By Theorem 1.1, these linear operators are continuous. The conjugate operators

$$L^*_{s,t} : H^{-1} \to L_2(D \times [s,t]), \quad \mathcal{L}^*_{s,t} : H^{-1} \to H^{-1}$$

are also linear and continuous.

Theorem 4.3 *Under the assumptions of Theorem 4.2 (with $D = \mathbf{R}^n$), the weak solution $y^{a,0}(t)$ of Eqn. (4.1) with $s = 0$ has the probability density function $p(\cdot,t) \in H^0$ for a.e. t. Moreover, $p \in L_2(Q)$, $p(\cdot,t) \in H^{-1}$ for all t, $p(\cdot,t) = \mathcal{L}^*_{0,t}\rho$ and $p = L^*_{0,T}\rho$ for the operators $\mathcal{L}^*_{0,t}$, $L^*_{0,T}$ defined for $\lambda \equiv 0$ (i.e., the probability density function $p(\cdot,t)$ is uniquely defined as an element of $L_2(Q)$ and is uniquely defined as an element of H^{-1}) for all t.*

Proof of Theorems 4.2–4.3. It suffices to consider $s = 0$.

(i) Let φ and Φ be such that

$$v \triangleq L\varphi + \mathcal{L}\Phi \in C^{2,1}(Q).$$

Here $L : X^0 \to Y^2$, $\mathcal{L} : H^1 \to Y^2$ are operators such that $v = L\varphi + \mathcal{L}\Phi$ is the solution of problem

$$\begin{cases} \frac{\partial v}{\partial t} + Av = -\varphi, \\ v(x,t)|_{x\in\partial D} = 0, \quad v(x,T) = \Phi(x) \end{cases} \tag{4.4}$$

(or the corresponding Cauchy problem for $D = \mathbf{R}^n$). In this case relation (4.3) follows from the Itô formula.

(ii) Let $\varphi \in X^0$ and $\Phi \in H^1$ be arbitrary. Introduce the sets

$$S_1 \triangleq \{\varphi \in X^0 : L\varphi \in C^{2,1}(Q)\}, \qquad S_2 \triangleq \{\Phi \in H^1 :\ \mathcal{L}\Phi \in C^{2,1}(Q)\}.$$

By Theorem 1.1, arbitrary functions $\varphi \in X^0$ and $\Phi \in H^1$ can be approximated in these spaces by $\varphi_\varepsilon \triangleq -\partial u^{(\varepsilon)}/\partial t - Au^{(\varepsilon)}$ and $\Phi_\varepsilon \triangleq u^{(\varepsilon)}(\cdot,T)$ respectively, where $u^{(\varepsilon)}$ is the Sobolev average of the functions $u = L\varphi$ or $u = \mathcal{L}\Phi$ respectively: by Theorem 1.1, $\varphi_\varepsilon \to \varphi$ in X^0 and $\Phi_\varepsilon \to \Phi$ in H^1 as $\varepsilon \to 0$. Hence, the sets S_1 and S_2 are dense in X^0 and in H^1, respectively.

Let $\bar{p} \triangleq L^*_{0,T}\rho$. This is an element of X^0, and $\bar{p}(\cdot,t) = \mathcal{L}^*_{0,t}\rho \in H^{-1}$ for all t. Let $p(x,t)$ be the probability density function of the process $y^{a,0}(t)$ being killed at ∂D if $D \neq \mathbf{R}^n$ and being killed inside D with the rate λ. The density $p(x,t)$ exists by the estimates from Section 2.3 from Krylov (1980). As was proved above for $\varphi \in S_1$ and $\Phi \in S_2$, we have

$$(v(\cdot,0),\rho)_{H^0} = (\varphi,p)_{X^0} + (p(\cdot,T),\Phi)_{H^0} = (\bar{p},\varphi)_{X^0} + (\bar{p}(\cdot,T),\Phi)_{H^0}.$$

Therefore, $p = \bar{p}$ and $p \in X^0$, $p(\cdot, T) = \bar{p}(\cdot, T) \in H^{-1}$.

Let $\varphi \in X^0$ and $\Phi \in H^1$ be arbitrary, and let $v \triangleq L\varphi + \mathcal{L}\Phi$. Let $v^{(\varepsilon)}$ be the Sobolev average of the function v in $\mathbf{R}^n \times \mathbf{R}$, let $\varphi_\varepsilon \triangleq -\partial v^{(\varepsilon)} \partial t - Av^{(\varepsilon)}$, and let $\Phi_\varepsilon \triangleq v^{(\varepsilon)}(\cdot, T)$. By Theorem 1.1, $\varphi_\varepsilon \to \varphi$ in X^0 and $\Phi_\varepsilon \to \Phi$ in H^1 as $\varepsilon \to 0$. We finally obtain the assertion of the theorem from the relation

$$\begin{aligned} (v(\cdot,0),\rho)_{H^0} = \lim_{\varepsilon\to 0}(v^{(\varepsilon)}(\cdot,0),\rho)_{H^0} \quad &= \lim_{\varepsilon\to 0}((\varphi_\varepsilon, p)_{X^0} + (p(\cdot,T),\Phi_\varepsilon)_{H^0}) \\ &= (p,\varphi)_{X^0} + (p(\cdot,T),\Phi)_{H^0} = F_{a,0}. \end{aligned}$$

□

Theorem 4.4 *Let a be a random vector, let $\mathbf{E}|a|^2 < +\infty$, and let ρ be the probability density function of a, $\rho \in H^{-1}$. Assume that Condition 4.1 is satisfied if f is replaced for $f \equiv 0$, an assume that the function f is measurable and bounded. Then problem (4.1) has a unique weak solution (i.e., the solution of (4.1) is univalent with respect to the probability distribution).*

Proof. It suffices to prove the uniqueness of the distribution of the process

$$z(t)^\top = [\operatorname{arctg} y_1^{a,0}(t), \dots, \operatorname{arctg} y_n^{a,0}(t)],$$

because the function arctg : $\mathbf{R} \to (-\pi, \pi)$ is one-to-one. We consider $z(t)$ as a generalized random process defined in Hida (1980) with the parameter space $L^2([0,T], \bar{\mathcal{B}}_1, \ell_1, \mathbf{R}^n)$. As is shown in Hida (1980), the distribution of the process $z(\cdot)$ is uniquely defined by the values of the functional

$$\widetilde{F}_{a,0}(\xi) \triangleq \mathbf{E} \exp\Big\{ - \int_0^T i\xi(t)^\top z(t) dt \Big\}.$$

on the set $\xi \in L^2([0,T], \bar{\mathcal{B}}_1, \ell_1, \mathbf{R}^n)$ or on the set of functions $C([0,T]; \mathbf{R}^n)$, which is dense in $L^2((0,T), \bar{\mathcal{B}}_1, \ell_1, \mathbf{R}^n)$. Here $i = \sqrt{-1}$.

It is easy to see that

$$\widetilde{F}_{a,0}(\xi) = 1 - i\mathbf{E} \int_0^T \xi(t)^\top z(t) \exp\Big\{ - \int_0^t i\xi(r)^{\mathrm{T}} z(r) dr \Big\} dt.$$

We first assume that $f \equiv 0$. By Theorem 4.2,

$$\widetilde{F}_{a,0}(\xi) = 1 - i(V, \rho)_{H^0},$$

where $V = L\varphi$ for

$$\varphi(x,t) \equiv \xi(t)^\top [\operatorname{arctg} x_1, \dots, \operatorname{arctg} x_n]^\top, \quad \lambda(x,t) \equiv i\varphi(x,t).$$

Hence $\widetilde{F}_{a,0}$ is unique for $\xi \in C((0,T);\mathbf{R}^n)$, and the weak solution is unique if $f \equiv 0$.

Let f be an arbitrary measurable bounded function. We apply Girsanov theorem. Consider the equation

$$\begin{cases} d\widetilde{y}(t) = \beta(\widetilde{y}(t),t)dw(t), \\ \widetilde{y}(0) = a. \end{cases}$$

As proved above, it has a unique weak solution. By Theorem 2 from Chapter 3 of Gihman and Skorohod (1975), the distribution of the solution $y^{a,0}(t)$ is uniquely determined by the distribution of $\widetilde{y}(t)$. Hence, the distribution of $y^{a,0}(t)$ is defined uniquely. This completes the proof. $\square$

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