UNIFORM EQUIPARTITION TEST BOUNDS
FOR MULTIPLY SEQUENCES

Marco Pollanen
Department of Mathematics
Trent University
Peterborough, ON K9J 7B8, CANADA
marcopollanen@trentu.ca

Abstract: For almost all \( x_0 \in [0,1) \), the multiply sequence \( x_n = ax_{n-1} \mod 1 \), with \( a > 1 \) an integer, is equidistributed. In this paper we show that equidistributed multiply sequences are not \( m \)-equipartitioned for any \( m > 2 \). We also provide uniform asymptotic bounds for equipartition tests for such sequences.

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1. Introduction

The sequence \( x_n = ax_{n-1} + c \mod 1 \), where \( a \) is a non-negative integer, is an important sequence that arises in number theory, fractals, and applied mathematics [4].

A sequence \( \langle y_n \rangle \) is equidistributed in \([0,1)\) if for all \( 0 \leq a < b \leq 1 \),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \chi_{[a,b)}(y_n) = b - a,
\]

where \( \chi_A \) is the characteristic function of a set \( A \).

For the case when \( a = 1 \), the sequence \( \langle x_n \rangle \) was studied by Weyl [7] and shown to be equidistributed if and only if \( c \) is irrational. In the case when \( a > 1 \), the sequence is referred to as a multiply sequence. It was first shown by Borel that, for \( a > 1 \) and \( c = 0 \), the sequence is equidistributed for almost all \( x_0 \).
As in [2], let \( \langle S_n \rangle \) be a sequence of propositions about the sequence \( \langle y_n \rangle \). We define
\[
P((S_n)) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} 1_{S_n \text{ is true}},
\]
when the limit exists.

A sequence \( \langle y_n \rangle \) is \( \infty \)-distributed [3] if, for every value of \( k \),
\[
P(a_1 \leq y_n < b_1, \ldots, a_k \leq y_{n+k-1} < b_k) = (b_1 - a_1) \cdots (b_k - a_k)
\]
for all real \( a_j, b_j \), with \( 0 \leq a_j < b_j \leq 1 \) for \( 1 \leq j \leq k \).

As an example, \( u_n = \theta^n \mod 1 \) is \( \infty \)-distributed for almost all real numbers \( \theta > 1 \) [2].

\( \infty \)-distributed sequences are of interest in that they automatically pass a
large number of asymptotic statistical tests, including: the frequency test, serial
test, gap test, poker test, coupon collector’s test, permutation test, run test,
maximum-of-t test, collision test, birthday spacings test, serial correlation test,
and tests on subsequences [3].

One test of particular importance is whether a sequence is \( m \)-equipartitioned.
A sequence \( \langle x_n \rangle \) is \( m \)-equipartitioned if for any permutation \( i_1, \ldots, i_m \) of the
index set \( \{i, \ldots, i + m - 1\} \) we have
\[
P(x_{i_1} > \ldots > x_{i_m}) = \frac{1}{m!}.
\]

Multiply sequences and their generalizations have been extensively studied
by Franklin [1, 2], and have been shown to be not \( 3 \)-equipartitioned. It was,
however, shown in [2] that, for almost all \( x_0 \), as \( a \to \infty \), multiply sequences
become \( \infty \)-distributed and \( m \)-equipartitioned, so in a sense multiply sequences
are almost \( \infty \)-distributed.

In the next section we demonstrate that for \( c = 0 \) and almost all \( x_0 \), multiply
sequences are only \( m \)-equipartitioned for \( m = 2 \). However, we also show that
for all \( m \), as \( a \to \infty \) they are uniformly \( m \)-equipartitioned by exhibiting an
explicit uniform bound. Thus, for large \( a \) we might expect multiply sequences
to have good statistical properties.

To prove these results we will use the concepts of generating functions (see
[8]) and tetrahedral numbers, which we will now briefly introduce.

A generating function for a sequences of real numbers \( \langle a_n \rangle \), \( n \geq 0 \) is an
infinite series \( G(x) = \sum_{n=0}^{\infty} a_n x^n \). This is a formal series, with questions about
convergence being ignored. Given a generating function, the sequence it repre-
sents can be determined by calculating the coefficients of \( x^n \), for all \( n \geq 0 \), in
its formal power series expansion.
The $n$-th triangular number is the sum $1 + 2 + \cdots + n$. It derives its name from the fact that the number can be represented graphically by a triangular lattice with $n$ rows, with the summands representing the number of points in each row.

The $n$-th tetrahedral number is the sum of the first $n$ triangular numbers, and can graphically be represented in three dimensions as an $n$-row tetrahedron, where the $i$-th row contains the $i$-th triangular number of points. Likewise, for $d > 3$ an integer, the $n$-th $d$-dimensional tetrahedral number will be the sum of the first $n$ $(d-1)$-dimensional tetrahedral numbers.

### 2. Uniform Bounds for Multiply Sequences

We will first use a generalization of a technique found in [2] to show that multiply sequences with $c = 0$ are not $m$-equipartitioned for $m > 2$.

**Theorem 2.1.** Let $x_n = ax_{n-1} \mod 1$ be an equidistributed sequence, with $a > 1$ an integer and $x_0 \in [0,1)$. Then

\[
P(x_i > x_{i+1} > \cdots > x_{i+k}) = \frac{1}{a^k(a-1)} \binom{a+k-1}{k+1},
\]

**Proof.** Assume $x_i > x_{i+1} > \cdots > x_{i+k}$. Now, for $i < j < i+k$, we have $x_j = \frac{A_j}{a} + \frac{x_{j+1}}{a}$ where $A_j$ are integers between 0 and $a-1$.

Clearly, if $A_j = 0$, then $x_{j+1} = ax_j > x_j$ for $a > 1$. Thus, $A_j \neq 0$.

Now, if $x_j > x_{j+1}$, then

\[
\frac{A_j}{a} + \frac{x_{j+1}}{a} > x_{j+1} \text{ or equivalently } x_{j+1} < \frac{A_j}{a-1}.
\]

If $A_{j+1} > A_j$, then

\[
x_j = \frac{A_j}{a} + \frac{x_{j+1}}{a} < \frac{A_j + 1}{a} \leq \frac{A_{j+1}}{a} \leq x_{j+1},
\]

which is a contradiction.

Thus, if $x_i > x_{i+1} > \cdots > x_{i+k}$, then

\[
a - 1 \geq A_i \geq A_{i+1} \geq \cdots \geq A_{i+k-1} \geq 1 \text{ and } 0 \leq x_{j+1} < \frac{A_j}{a-1}. \tag{1}
\]

Conversely, the inequalities (1) imply

\[
x_{j+1} < \frac{A_j}{a} + \frac{x_{j+1}}{a} = x_j,
\]
and so $x_i > x_{i+1} > \ldots > x_{i+k}$.

Now, for each $x_i \in [0, 1)$, we have the unique $a$-ary representation

$$x_i = \frac{A_i}{a} + \frac{A_{i+1}}{a^2} + \cdots + \frac{A_{i+k-1}}{a^k} + \frac{x_{i+k}}{a^k}.$$ 

For each possible set $\{A_i, \ldots, A_{i+k-1}\}$, the length of the interval of permissible $x_i$’s is $\frac{A_{i+k-1}}{a-1}$. Hence, the measure of the set of $x_i$’s is:

$$\sum_{A_i=1}^{a-1} \sum_{A_{i+1}=1}^{A_i} \cdots \sum_{A_{i+k-1}=1}^{A_{i+k-2}} \frac{1}{a^k} = \frac{1}{a^k(a-1)} \sum_{A_i=1}^{a-1} \sum_{A_{i+1}=1}^{A_i} \cdots \sum_{A_{i+k}=1}^{A_{i+k-1}} 1.$$ 

Ignoring the factor in front, the last sum is a tetrahedral number, so combinatorially the sum is

$$\binom{a+k-1}{k},$$

and so we are done. This follows since triangular numbers have a generating function $x/(1-x)^3$ [6] and, by convolution, the $d$-dimensional tetrahedral numbers have a generating function $x/(1-x)^{d+1}$. Accordingly, the $n$-th $d$-tetrahedral number can be combinatorially represented as

$$T_d(n) = \sum_{A_i=1}^{n} \sum_{A_2=1}^{A_i} \cdots \sum_{A_d=1}^{A_{d-1}} 1 = \binom{n+d-1}{d}.$$  \hspace{1cm} (2)

From this theorem, the following corollary of [2] is immediate. 

**Corollary 1.** Let $x_n = ax_{n-1} \mod 1$ be an equidistributed sequence, with $a > 1$ an integer and $x_0 \in [0, 1)$. Then $P(x_i > x_{i+1}) = P(x_i < x_{i+1}) = \frac{1}{2}$

and $P(x_i > x_{i+1} > x_{i+2}) = \frac{1}{3!} + \frac{1}{6a}$. 

The multiply sequence is not 3-equipartitioned. However, as the next corollary demonstrates, in a sense the case of 3-equipartitioning is the worst case.
Corollary 2. Let \( x_n = ax_{n-1} \mod 1 \) be an equidistributed sequence, with \( a > 1 \) an integer and \( x_0 \in [0, 1) \). Then

\[
\left| P(x_i > x_{i+1} > \cdots > x_{i+k}) - \frac{1}{(k+1)!} \right| \leq \frac{1}{6a},
\]

Proof. We have equality for \( k = 2 \). Accordingly, we will proceed by induction, by assuming

\[
\frac{1}{a^k(a-1)} \left( \frac{a+k-1}{k+1} \right) \leq \frac{1}{(k+1)!} + \frac{1}{6a},
\]

that is,

\[
\frac{6(a+k-1)!}{a!} \leq 6a^{k-1} + a^{k-2}(k+1)!.\]

Now, as \( a \geq 2 \), \( 6a \leq (a-1)(k+1)! + a(k+1)(k-1)! \) holds for \( k = 2 \). It also holds for \( k \geq 2 \). Thus, if we expand the first term on the right hand side and multiplying both sides by \( ka^{k-1} \), we have

\[
6ka^{k-1} + ka^{k-2}(k+1)! \leq ka^{k-1}(k+1)! + a^{k-1}(k+1)! = a^{k-1}(k+2)! - a^{k-1}(k+1)!.
\]

Thus, using our induction hypothesis:

\[
\frac{6(a+k)!}{a!} \leq (a+k)(6a^{k-1} + a^{k-2}(k+1)!)
\]

\[
= 6a^k + a^{k-1}(k+1)! + 6ka^{k-1} + ka^{k-2}(k+1)!
\]

\[
\leq 6a^k + a^{k-1}(k+2)!.
\]

Hence, by induction, the desired result holds for all \( a, k \geq 2 \).

We now show that, for all possible equipartition tests, the error in the estimate in the last corollary is uniformly bounded as \( a \to \infty \).

Theorem 2.2. Let \( x_n = ax_{n-1} \mod 1 \) be an equidistributed sequence, with \( a > 1 \) an integer and \( x_0 \in [0, 1) \). Then, for any permutation \( i_1, \ldots, i_m \) of the index set \( \{1, \ldots, i + m - 1\} \), we have

\[
\frac{3}{a} \leq P(x_{i_1} > x_{i_2} > \cdots > x_{i_m}) - \frac{1}{m!} \leq \frac{3}{a - 1}.
\]

Proof. Suppose \( x_0 \in [0, 1) \) and \( x_n = ax_{n-1} \mod 1 \), with \( a > 1 \) an integer, is an equidistributed sequence. We may write \( x_i \) with a unique \( a \)-ary expansion

\[
x_i = \frac{A_1}{a} + \frac{A_2}{a^2} + \cdots + \frac{A_k}{a^k} + \frac{x_{i+k}}{a^k},
\]
where $0 \leq A_j < a$ for all $j \in \{1, \ldots, k\}$. Now it follows that

\[
x_{i+1} = \frac{A_2}{a} + \frac{A_3}{a^2} + \cdots + \frac{A_k}{a^{k-1}} + \frac{x_{i+k}}{a^{k-1}}
\]

\[
\vdots
\]

\[
x_{i+j} = \frac{A_{i+j}}{a} + \frac{A_{i+j+2}}{a^2} + \cdots + \frac{A_k}{a^{k-j}} + \frac{x_{i+k}}{a^{k-j}}
\]

\[
\vdots
\]

\[
x_{i+k-1} = \frac{A_k}{a} + \frac{x_{i+k}}{a}
\]

Let us define a reordering of the index set $\{i, \ldots, i+k-1\}$, say $\{j_1, \ldots, j_k\}$, such that

\[
x_{j_1} \geq x_{j_2} \geq \cdots \geq x_{j_k}
\]  

(4)

and, for some $l \in \{1, \ldots, k-1\}$,

\[
x_{j_l} > x_{i+k} > x_{j_{l+1}}
\]  

(5)

The last assumption (5) is for notational convenience. The following calculation could be easily repeated, and is in fact simpler, with $x_{i+k}$ being either the smallest or largest of the consecutive members of the sequence.

Now, if we compare the first terms in the expansions of $x_{j_1}, \ldots, x_{j_k}$, it follows that

\[
a - 1 \geq A_{j_1} \geq A_{j_2} \geq \cdots \geq A_{j_k} \geq 0,
\]  

(6)

and so by inequality (5) we have

\[
\frac{A_{j_1}}{a} + \frac{A_{j_1+1}}{a^2} + \cdots + \frac{A_k}{a^{k-j_l+1}} + \frac{x_{i+k}}{a^{k-j_l+1}} > x_{i+k}
\]

\[
> \frac{A_{j_1}}{a} + \frac{A_{j_1+1}}{a^2} + \cdots + \frac{A_k}{a^{k-j_l+1}} + \frac{x_{i+k}}{a^{k-j_l+1}}.
\]

If we write this using common denominators and isolate $x_{i+k}$, we have

\[
\frac{A_j a^{k-j_l} + \cdots + A_k}{a^{k-j_l+1} - 1} > x_{i+k} > \frac{A_{j_1} a^{k-j_l+1} + \cdots + A_k}{a^{k-j_l+1} - 1}.
\]  

(7)

We wish to calculate the measure of the set of initial values $x_i$ that give us the ordering (4) and (5). To do this, for each combination of $A_j$’s, we will establish upper and lower estimates for the length of the interval of $x_{i+k}$’s that satisfy the inequality (7). Thus, we want to calculate

\[
I(A_1, \ldots, A_k) = \frac{A_j a^{k-j_l} + \cdots + A_k}{a^{k-j_l+1} - 1} + \frac{A_{j_1} a^{k-j_l+1} + \cdots + A_k}{a^{k-j_l+1} - 1},
\]
which we will abbreviate as $I$.

Note that for integers $a > 1$ and $b > 1$

$$
\frac{1}{a} < \frac{a^b}{a^{b+1} - 1} < \frac{1}{a-1},
$$

and so

$$
I \geq \frac{A_{j_1}a^{k-j_1}}{a^{k-j_1+1} - 1} - \frac{A_{j_{i+1}}a^{k-j_{i+1}} + (a-1)a^{k-j_{i+1}-1}\cdots + (a-1)}{a^{k-j_{i+1}+1} - 1}
$$

$$
= \frac{A_{j_1}a^{k-j_1}}{a^{k-j_1+1} - 1} - \frac{A_{j_{i+1}}a^{k-j_{i+1}}}{a^{k-j_{i+1}+1} - 1} - \frac{(a-1)(a^{k-j_{i+1}} - 1)}{(a-1)a^{k-j_{i+1}+1} - 1}
$$

$$
> \frac{A_{j_1}a^{k-j_1}}{a^{k-j_1+1} - 1} - \frac{(A_{j_{i+1}} + 1)a^{k-j_{i+1}}}{a^{k-j_{i+1}+1} - 1}
$$

$$
> \frac{A_{j_1}a^{k-j_1}}{a} - \frac{A_{j_{i+1}} + 1}{a-1}.
$$

Similarly,

$$
I < \frac{A_{j_1} + 1}{a} - \frac{A_{j_{i+1}}}{a-1}.
$$

We will now establish lower and upper bounds for

$$
P(x_{j_1} > \ldots > x_{j_i} > x_{i+k} > x_{j_{i+1}} > \ldots > x_{j_k}),
$$

which we will abbreviate as $P$.

From equation (3) we see that we can find an upper bound for (8) by calculating $I(A_1, \ldots, A_k)$ for each possible combination of $A_1, \ldots, A_k$, such that inequalities (6) are satisfied, that is

$$
P \leq \sum_{A_{j_1}=0}^{a-1} \sum_{A_{j_2}=0}^{A_{j_1}} \cdots \sum_{A_{j_k}=0}^{A_{j_{k-1}}} \frac{I(A_1, \ldots, A_k)}{a^k}
$$

$$
< \sum_{A_{j_1}=0}^{a-1} \sum_{A_{j_2}=0}^{A_{j_1}} \cdots \sum_{A_{j_k}=0}^{A_{j_{k-1}}} \frac{1}{a^k} \left( \frac{A_{j_1} + 1}{a} - \frac{A_{j_{i+1}}}{a-1} \right).
$$

For a lower bound, we note that if

$$
a - 1 > A_{j_1} > A_{j_2} > \ldots > A_{j_k} > 0,
$$
then surely \( x_{j_1} > \ldots > x_{j_k} \). Hence, we can establish the following lower bound:

\[
P \geq \sum_{A_{j_1} = 1}^{a-1} \sum_{A_{j_2} = 1}^{A_{j_1}-1} \cdots \sum_{A_{j_k} = 1}^{A_{j_{k-1}}-1} \frac{I(A_1, \ldots, A_k)}{a^k} \sum_{A_{j_1} = 1}^{a-1} \sum_{A_{j_2} = 1}^{A_{j_1}-1} \cdots \sum_{A_{j_k} = 1}^{A_{j_{k-1}}-1} \frac{1}{a^k} \left( \frac{A_{j_k} - A_{j_{k+1}} + 1}{a} \right).
\]

Thus, the key to estimating \( P \) is to calculate sums of the form

\[
\sum_{B_1 = 0}^{n} \sum_{B_2 = 0}^{B_1} \cdots \sum_{B_d = 0}^{B_{d-1}} B_{d-m+1} \tag{11}
\]

and

\[
\sum_{B_1 = 1}^{n-1} \sum_{B_2 = 1}^{B_1-1} \cdots \sum_{B_d = 1}^{B_{d-1}-1} B_{d-m+1} \tag{12}
\]

for integers \( m \), \( 1 \leq m \leq d \).

The sum

\[
\sum_{B_1 = 0}^{n} \sum_{B_2 = 0}^{B_1} \cdots \sum_{B_n = 0}^{1}
\]

is the coefficient of \( x^n \) in the generating function \( 1/(1-x)^{d+1} \) or, in other words, it is given by \( T_d(n+1) \).

Thus, the sum in (11) can be represented by the generating function

\[
\frac{x}{(1-x)^{d-m+1}} \frac{d}{dx} \left( \frac{1}{(1-x)^m} \right) = \frac{mx}{(1-x)^{d+2}}
\]

and so,

\[
\sum_{B_1 = 0}^{n} \sum_{B_2 = 0}^{B_1} \cdots \sum_{B_d = 0}^{B_{d-1}} B_{d-m+1} = m T_{d+1}(n) = m \left( \frac{n + d}{d + 1} \right).
\]

Likewise,

\[
\sum_{B_1 = 1}^{n-1} \sum_{B_2 = 1}^{B_1-1} \cdots \sum_{B_d = 1}^{B_{d-1}-1} B_{d-m+1} = m T_{d+1}(n - d) = m \left( \frac{n}{d + 1} \right).
\]
We will now show that $P$ positive. By substituting $a = 1$, we see that they sum up to $(1+2k)! < 2(k+1)!$. Thus,

\[ U > \frac{a^k(a-1) + 3a^k(k+1)! - (a^{k+1} + (2k + \sum_{i=1}^{k-1} i)a^k + 2a^k(k+1)!)}{a^k(a-1)(k+1)!} \]

Likewise, from (10) we have the following lower bound for (8):

\[ \frac{k-l+1}{a^{k+1}(a-1)} \left( \frac{a}{k+1} \right) - \frac{k-l}{a^k(a-1)} \left( \frac{a}{k+1} \right) - \frac{1}{a^k(a-1)} \left( \frac{a}{k} \right) \]

Thus, we have shown that

\[ \frac{a-2k-2}{a^{k+1}(a-1)} \left( \frac{a}{k+1} \right) < P < \frac{a+2k}{a^k(a-1)^2} \left( \frac{a+k-1}{k+1} \right). \]

We will now show that $P$ has uniform bounds in terms of $a$ and $k$, for all $a \geq 2$ and $k \geq 2$.

\[ \frac{1}{(k+1)!} - \frac{3}{a} < P < \frac{1}{(k+1)!} + \frac{3}{a-1}. \]

Consider

\[ U = \frac{1}{(k+1)!} + \frac{3}{a-1} - \frac{a+2k}{a^k(a-1)^2} \left( \frac{a+k-1}{k+1} \right) \]

\[ = \frac{a^k(a-1) + 3a^k(k+1)! - (a^{k+1} + (2k + \sum_{i=1}^{k-1} i)a^k + 2a^k(k+1)!)}{a^k(a-1)(k+1)!} \]

Note that all the coefficients in the polynomial $f(a) = (a+2k)(a+k-1) \cdots a$ are positive. By substituting $a = 1$, we see that they sum up to $(1+2k)! < 2(k+1)!$. Thus,

\[ U > \frac{a^k(a-1) + 3a^k(k+1)! - (a^{k+1} + (2k + \sum_{i=1}^{k-1} i)a^k + 2a^k(k+1)!)}{a^k(a-1)(k+1)!} \]
\[
\frac{a^k(k+1)! - 0.5(k+1)(k+2)a^k}{a^k(a-1)(k+1)!} \geq 0.
\]

The last inequality follows as \(2(k+1)! \geq (k+1)(k+2)\), for \(k \geq 2\).

Now, consider
\[
L = \frac{a - 2k - 2}{a^{k+1}(a-1)} \left( \frac{a}{k+1} \right) + \frac{3}{a} - \frac{1}{(k+1)!}
\]
\[
= \frac{(a - 2(k+1))(a-2) \cdots (a-k) + 3a^{k-1}(k+1)! - a^k}{a^k(k+1)!}.
\]

Consider the expansion of the polynomial \(f(a) = (a-2(k+1))(a-2) \cdots (a-k)\)
\[
f(a) = a^k - \left( 2(k+1) + \sum_{2 \leq i \leq k} i \right) a^{k-1} + \]
\[
+ \left( 2(k+1) \sum_{2 \leq i \leq k} i + \sum_{2 \leq i < j \leq k} ij \right) a^{k-2} - \cdots + (-1)^k 2(k+1)!.\]

In the expansion of the polynomial \((a-2) \cdots (a-k)\), the absolute value of
the coefficient of \(a^r\) will not exceed
\[
\left( \binom{k-1}{r} \right) r! = \frac{(k-1)!k!}{r!(k-r-1)!} \leq (k-1)!k!.
\]

Since, for \(a \leq (k+1)!\), \(L\) is trivially positive, we consider \(a \geq (k+1)!\). In this case, we see that, in the expansion of the polynomial \(f(a)\), the absolute value of the term involving \(a^r\), for \(k > 1\), will not exceed
\[
(2(k+1)(k-1)!k! + (k-1)!k!) a^r < 2(k+1)!k!a^r < 2k!a^{r+1}.
\]

Now, discarding the positive coefficients of \(a^r\), for \(r < k - 1\) and bounding the \(\left\lfloor \frac{k}{2} \right\rfloor - 1 \leq \frac{k}{2}\) negative terms by \(-2k!a^{k-1}\), we have
\[
f(a) > a^k - 0.5(k^2 + 5k + 2)a^{k-1} - kk!a^{k-1}.
\]

However, for \(k \geq 2\), \(2(k+1)! > 0.5(k^2 + 5k + 2)\) and so, \(f(a) > a^k - 3(k+1)!a^{k-1}\).

Thus,
\[
L > \frac{a^k - 3(k+1)!a^{k-1} + 3a^{k-1}(k+1)! - a^k}{a^k(k+1)!} = 0.
\]

Thus, the desired result follows. \(\square\)
3. Discussion

In Theorem 2.1 we demonstrated that equidistributed multiply sequences are not $m$-equipartitioned for any $m > 2$ by establishing an exact value for $P(x_i > x_{i+1} > \cdots > x_{i+m-1})$. Calculations for other individual permutations can be established in a similar fashion. For example, for $m = 3$ it is easy to show that:

$$P(x_i > x_{i+1} > x_{i+2}) = P(x_{i+2} > x_{i+1} > x_i) = \frac{1}{6} \left( 1 + \frac{1}{a} \right),$$

$$P(x_{i+1} > x_i > x_{i+2}) = P(x_{i+2} > x_i > x_{i+1}) = \frac{1}{6} \left( 1 - \frac{1}{a} \right),$$

and

$$P(x_i > x_{i+2} > x_{i+1}) = P(x_{i+1} > x_{i+2} > x_i) = \frac{1}{6}.$$

Exact values for $m > 3$ can likewise be established, although they become increasingly more complex.

However, in Theorem 2.2 we established a uniform bound for all equipartition tests of equidistributed multiply sequences. As multiply sequences are almost $\infty$-distributed in an asymptotic sense, these bounds can be thought of as one measure of how close a sequence is to $\infty$-distribution.

In comparing the general bounds of Theorem 2.2 to the above permutation tests for $m = 3$ it is apparent that the bounds can be improved. We have provided some alternate bounds, such as inequality (14), in intermediary steps. For large $a$ and $m$, other bounds might be possible from our calculations by using Stirling-like approximations [5].

We conjecture in fact that Theorem 2.1 provides the worst case upper bound, and a symmetric worst case lower bound, as follows:

**Conjecture 1.** Let $x_n = ax_{n-1} \mod 1$ be an equidistributed sequence, with $a > 1$ an integer and $x_0 \in [0, 1)$. Then, for any permutation $i_0, \ldots, i_k$ of the index set \{i, \ldots, i+k\}, we have

$$\left| P(x_{i_0} > x_{i_2} > \cdots > x_{i_k}) - \frac{1}{(k+1)!} \right| \leq \frac{1}{a^k(a-1)} \binom{a+k-1}{k+1} - \frac{1}{(k+1)!}.$$

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References


