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UNIFORM EQUIPARTITION TEST BOUNDS FOR MULTIPLY SEQUENCES

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Abstract: For almost all $x_0 \in [0,1)$, the multiply sequence $x_n = ax_{n-1} \bmod 1$, with $a > 1$ an integer, is equidistributed. In this paper we show that equidistributed multiply sequences are not m-equipartitioned for any $m > 2$. We also provide uniform asymptotic bounds for equipartition tests for such sequences.

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1. Introduction

The sequence $x_n = ax_{n-1} + c \bmod 1$, where a is a non-negative integer, is an important sequence that arises in number theory, fractals, and applied mathematics [4].

A sequence $\langle y_n \rangle$ is equidistributed in $[0,1)$ if for all $0 \le a < b \le 1$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\le n\le N} \chi_{[a,b)}(y_n) = b - a,$$

where χ_A is the characteristic function of a set A.

For the case when $a = 1$, the sequence $\langle x_n \rangle$ was studied by Weyl [7] and shown to be equidistributed if and only if c is irrational. In the case when $a > 1$, the sequence is referred to as a multiply sequence. It was first shown by Borel that, for $a > 1$ and $c = 0$, the sequence is equidistributed for almost all x_0.

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As in [2], let $\langle S_n \rangle$ be a sequence of propositions about the sequence $\langle y_n \rangle$. We define

$$P(\langle S_n \rangle) = \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{S_n \text{ is true} \\ 1\le n\le N}} 1 \quad ,$$

when the limit exists.

A sequence $\langle y_n \rangle$ is ∞-distributed [3] if, for every value of k,

$$P(a_1 \le y_n < b_1, \cdots, a_k \le y_{n+k-1} < b_k) = (b_1 - a_1)\cdots(b_k - a_k)$$

for all real a_j, b_j, with $0 \le a_j < b_j \le 1$ for $1 \le j \le k$.

As an example, $u_n = \theta^n \bmod 1$ is ∞-distributed for almost all real numbers $\theta > 1$ [2].

∞-distributed sequences are of interest in that they automatically pass a large number of asymptotic statistical tests, including: the frequency test, serial test, gap test, poker test, coupon collector's test, permutation test, run test, maximum-of-t test, collision test, birthday spacings test, serial correlation test, and tests on subsequences [3].

One test of particular importance is whether a sequence is m-equipartitioned. A sequence $\langle x_n \rangle$ is m-equipartitioned if for any permutation $i_1, \ldots, i_m$ of the index set $\{i, \ldots, i+m-1\}$ we have

$$P(x_{i_1} > \ldots > x_{i_m}) = \frac{1}{m!}.$$

Multiply sequences and their generalizations have been extensively studied by Franklin [1, 2], and have been shown to be not 3-equipartitioned. It was, however, shown in [2] that, for almost all x_0, as $a \to \infty$, multiply sequences become ∞-distributed and m-equipartitioned, so in a sense multiply sequences are *almost* ∞-distributed.

In the next section we demonstrate that for $c = 0$ and almost all x_0, multiply sequences are only m-equipartitioned for $m = 2$. However, we also show that for all m, as $a \to \infty$ they are uniformly m-equipartitioned by exhibiting an explicit uniform bound. Thus, for large a we might expect multiply sequences to have good statistical properties.

To prove these results we will use the concepts of generating functions (see [8]) and tetrahedral numbers, which we will now briefly introduce.

A generating function for a sequences of real numbers $\langle a_n \rangle$, $n \ge 0$ is an infinite series $G(x) = \sum_{n=0}^{\infty} a_n x^n$. This is a formal series, with questions about convergence being ignored. Given a generating function, the sequence it represents can be determined by calculating the coefficients of x^n, for all $n \ge 0$, in its formal power series expansion.

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The n-th triangular number is the sum $1 + 2 + \cdots + n$. It derives its name from the fact that the number can be represented graphically by a triangular lattice with n rows, with the summands representing the number of points in each row.

The n-th tetrahedral number is the sum of the first n triangular numbers, and can graphically be represented in three dimensions as an n-row tetrahedron, where the i-th row contains the i-th triangular number of points. Likewise, for $d > 3$ an integer, the n-th d-dimensional tetrahedral number will be the sum of the first n $(d-1)$-dimensional tetrahedral numbers.

2. Uniform Bounds for Multiply Sequences

We will first use a generalization of a technique found in [2] to show that multiply sequences with $c = 0$ are not m-equipartitioned for $m > 2$.

Theorem 2.1. *Let $x_n = ax_{n-1}$ mod 1 be an equidistributed sequence, with $a > 1$ an integer and $x_0 \in [0, 1)$. Then*

$$P(x_i > x_{i+1} > \cdots > x_{i+k}) = \frac{1}{a^k(a-1)} \binom{a+k-1}{k+1}.$$

Proof. Assume $x_i > x_{i+1} > \cdots > x_{i+k}$. Now, for $i < j < i + k$, we have $x_j = \frac{A_j}{a} + \frac{x_{j+1}}{a}$ where A_j are integers between 0 and $a - 1$.

Clearly, if $A_j = 0$, then $x_{j+1} = ax_j > x_j$ for $a > 1$. Thus, $A_j \neq 0$.

Now, if $x_j > x_{j+1}$, then

$$\frac{A_j}{a} + \frac{x_{j+1}}{a} > x_{j+1} \text{ or equivalently } x_{j+1} < \frac{A_j}{a-1}.$$

If $A_{j+1} > A_j$, then

$$x_j = \frac{A_j}{a} + \frac{x_{j+1}}{a} < \frac{A_j + 1}{a} \leq \frac{A_{j+1}}{a} \leq x_{j+1},$$

which is a contradiction.

Thus, if $x_i > x_{i+1} > \ldots > x_{i+k}$, then

$$a - 1 \geq A_i \geq A_{i+1} \geq \ldots \geq A_{i+k-1} \geq 1 \text{ and } 0 \leq x_{j+1} < \frac{A_j}{a-1}. \quad (1)$$

Conversely, the inequalities (1) imply

$$x_{j+1} < \frac{A_j}{a} + \frac{x_{j+1}}{a} = x_j,$$

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and so $x_i > x_{i+1} > \ldots > x_{i+k}$.

Now, for each $x_i \in [0,1)$, we have the unique a-ary representation

$$x_i = \frac{A_i}{a} + \frac{A_{i+1}}{a^2} + \cdots + \frac{A_{i+k-1}}{a^k} + \frac{x_{i+k}}{a^k}.$$

For each possible set $\{A_i, \ldots, A_{i+k-1}\}$, the length of the interval of permissible x_i's is $\frac{A_{i+k-1}}{a-1}$. Hence, the measure of the set of x_i's is:

$$\sum_{A_i=1}^{a-1} \sum_{A_{i+1}=1}^{A_i} \cdots \sum_{A_{i+k-1}=1}^{A_{i+k-2}} \frac{1}{a^k} \frac{A_{i+k-1}}{a-1}$$

$$= \frac{1}{a^k(a-1)} \sum_{A_i=1}^{a-1} \sum_{A_{i+1}=1}^{A_i} \cdots \sum_{A_{i+k}=1}^{A_{i+k-1}} 1.$$

Ignoring the factor in front, the last sum is a tetrahedral number, so combinatorially the sum is

$$\binom{a+k-1}{k+1},$$

and so we are done. This follows since triangular numbers have a generating function $x/(1-x)^3$ [6] and, by convolution, the d-dimensional tetrahedral numbers have a generating function $x/(1-x)^{d+1}$. Accordingly, the n-th d-tetrahedral number can be combinatorially represented as

$$T_d(n) = \sum_{A_1=1}^{n} \sum_{A_2=1}^{A_1} \cdots \sum_{A_d=1}^{A_{d-1}} 1 = \binom{n+d-1}{d}. \tag{2}$$

□

From this theorem, the following corollary of [2] is immediate.

Corollary 1. *Let $x_n = ax_{n-1}$ mod 1 be an equidistributed sequence, with $a > 1$ an integer and $x_0 \in [0,1)$. Then*

$$P(x_i > x_{i+1}) = P(x_i < x_{i+1}) = \frac{1}{2}$$

$$\text{and } P(x_i > x_{i+1} > x_{i+2}) = \frac{1}{3!} + \frac{1}{6a}.$$

The multiply sequence is not 3-equipartitioned. However, as the next corollary demonstrates, in a sense the case of 3-equipartitioning is the worst case.

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Corollary 2. *Let* $x_n = ax_{n-1}$ *mod 1 be an equidistributed sequence, with* $a > 1$ *an integer and* $x_0 \in [0,1)$. *Then*

$$\left| P(x_i > x_{i+1} > \cdots > x_{i+k}) - \frac{1}{(k+1)!} \right| \leq \frac{1}{6a}.$$

Proof. We have equality for $k = 2$. Accordingly, we will proceed by induction, by assuming

$$\frac{1}{a^k(a-1)} \begin{pmatrix} a+k-1 \\ k+1 \end{pmatrix} \leq \frac{1}{(k+1)!} + \frac{1}{6a},$$

that is,

$$\frac{6(a+k-1)!}{a!} \leq 6a^{k-1} + a^{k-2}(k+1)!.$$

Now, as $a \geq 2$, $6a \leq (a-1)(k+1)! + a(k+1)(k-1)!$ holds for $k = 2$. It also holds for $k \geq 2$. Thus, if we expand the first term on the right hand side and multiplying both sides by ka^{k-2}, we have

$$6ka^{k-1} + ka^{k-2}(k+1)! \leq ka^{k-1}(k+1)! + a^{k-1}(k+1)! = a^{k-1}(k+2)! - a^{k-1}(k+1)!.$$

Thus, using our induction hypothesis:

$$\begin{aligned} 6\frac{(a+k)!}{a!} &\leq (a+k)(6a^{k-1} + a^{k-2}(k+1)!) \\ &= 6a^k + a^{k-1}(k+1)! + 6ka^{k-1} + ka^{k-2}(k+1)! \\ &\leq 6a^k + a^{k-1}(k+2)!. \end{aligned}$$

Hence, by induction, the desired result holds for all $a, k \geq 2$. □

We now show that, for all possible equipartition tests, the error in the estimate in the last corollary is uniformly bounded as $a \to \infty$.

Theorem 2.2. *Let* $x_n = ax_{n-1}$ *mod 1 be an equidistributed sequence, with* $a > 1$ *an integer and* $x_0 \in [0,1)$. *Then, for any permutation* $i_1, \ldots, i_m$ *of the index set* $\{i, \ldots, i+m-1\}$, *we have*

$$-\frac{3}{a} \leq P(x_{i_1} > x_{i_2} > \cdots > x_{i_m}) - \frac{1}{m!} \leq \frac{3}{a-1}.$$

Proof. Suppose $x_0 \in [0,1)$ and $x_n = ax_{n-1}$ mod 1, with $a > 1$ an integer, is an equidistributed sequence. We may write x_i with a unique a-ary expansion

$$x_i = \frac{A_1}{a} + \frac{A_2}{a^2} + \cdots + \frac{A_k}{a^k} + \frac{x_{i+k}}{a^k}, \tag{3}$$

where $0 \le A_j < a$ for all $j \in \{1, \ldots, k\}$. Now it follows that

$$\begin{aligned} x_{i+1} &= \frac{A_2}{a} + \frac{A_3}{a^2} + \cdots + \frac{A_k}{a^{k-1}} + \frac{x_{i+k}}{a^{k-1}} \\ &\vdots \\ x_{i+j} &= \frac{A_{j+1}}{a} + \frac{A_{j+2}}{a^2} + \cdots + \frac{A_k}{a^{k-j}} + \frac{x_{i+k}}{a^{k-j}} \\ &\vdots \\ x_{i+k-1} &= \frac{A_k}{a} + \frac{x_{i+k}}{a}. \end{aligned}$$

Let us define a reordering of the index set $\{i, \ldots, i+k-1\}$, say $\{j_1, \ldots, j_k\}$ such that

$$x_{j_1} \ge x_{j_2} \ge \ldots \ge x_{j_k} \tag{4}$$

and, for some $l \in \{1, \ldots, k-1\}$,

$$x_{j_l} > x_{i+k} > x_{j_{l+1}}. \tag{5}$$

The last assumption (5) is for notational convenience. The following calculation could be easily repeated, and is in fact simpler, with x_{i+k} being either the smallest or largest of the consecutive members of the sequence.

Now, if we compare the first terms in the expansions of $x_{j_1}, \ldots, x_{j_k}$, it follows that

$$a - 1 \ge A_{j_1} \ge A_{j_2} \ge \ldots \ge A_{j_k} \ge 0, \tag{6}$$

and so by inequality (5) we have

$$\begin{aligned} \frac{A_{j_l}}{a} + \frac{A_{j_l+1}}{a^2} + \cdots + \frac{A_k}{a^{k-j_l+1}} + \frac{x_{i+k}}{a^{k-j_l+1}} > x_{i+k} \\ > \frac{A_{j_{l+1}}}{a} + \frac{A_{j_{l+1}+1}}{a^2} + \cdots + \frac{A_k}{a^{k-j_{l+1}+1}} + \frac{x_{i+k}}{a^{k-j_{l+1}+1}}. \end{aligned}$$

If we write this using common denominators and isolate x_{i+k}, we have

$$\frac{A_{j_l} a^{k-j_l} + \cdots + A_k}{a^{k-j_l+1} - 1} > x_{i+k} > \frac{A_{j_{l+1}} a^{k-j_{l+1}} + \cdots + A_k}{a^{k-j_{l+1}+1} - 1}. \tag{7}$$

We wish to calculate the measure of the set of initial values x_i that give us the ordering (4) and (5). To do this, for each combination of A_j's, we will establish upper and lower estimates for the length of the interval of x_{i+k}'s that satisfy the inequality (7). Thus, we want to calculate

$$I(A_1, \ldots, A_k) = \frac{A_{j_l} a^{k-j_l} + \cdots + A_k}{a^{k-j_l+1} - 1} - \frac{A_{j_{l+1}} a^{k-j_{l+1}} + \cdots + A_k}{a^{k-j_{l+1}+1} - 1},$$

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which we will abbreviate as I.

Note that for integers $a > 1$ and $b > 1$

$$\frac{1}{a} < \frac{a^b}{a^{b+1}-1} < \frac{1}{a-1},$$

and so

$$\begin{aligned} I &\geq \frac{A_{j_l}a^{k-j_l}}{a^{k-j_l+1}-1} - \frac{A_{j_{l+1}}a^{k-j_{l+1}} + (a-1)a^{k-j_{l+1}-1}\cdots + (a-1)}{a^{k-j_{l+1}+1}-1} \\ &= \frac{A_{j_l}a^{k-j_l}}{a^{k-j_l+1}-1} - \frac{A_{j_{l+1}}a^{k-j_{l+1}}}{a^{k-j_{l+1}+1}-1} - \frac{(a-1)(a^{k-j_{l+1}}-1)}{(a-1)a^{k-j_{l+1}+1}-1} \\ &> \frac{A_{j_l}a^{k-j_l}}{a^{k-j_l+1}-1} - \frac{(A_{j_{l+1}}+1)a^{k-j_{l+1}}}{a^{k-j_{l+1}+1}-1} \\ &> \frac{A_{j_l}}{a} - \frac{A_{j_{l+1}}+1}{a-1}. \end{aligned}$$

Similarly,

$$I < \frac{A_{j_l}+1}{a-1} - \frac{A_{j_{l+1}}}{a}.$$

We will now establish lower and upper bounds for

$$P(x_{j_1} > \ldots > x_{j_l} > x_{i+k} > x_{j_{l+1}} > \ldots > x_{j_k}), \tag{8}$$

which we will abbreviate as P.

From equation (3) we see that we can find an upper bound for (8) by calculating $I(A_1, \ldots, A_k)$ for each possible combination of $A_1, \cdots, A_k$, such that inequalities (6) are satisfied, that is

$$\begin{aligned} P &\leq \sum_{A_{j_1}=0}^{a-1} \sum_{A_{j_2}=0}^{A_{j_1}} \cdots \sum_{A_{j_k}=0}^{A_{j_{k-1}}} \frac{I(A_1, \ldots, A_k)}{a^k} \\ &< \sum_{A_{j_1}=0}^{a-1} \sum_{A_{j_2}=0}^{A_{j_1}} \cdots \sum_{A_{j_k}=0}^{A_{j_{k-1}}} \frac{1}{a^k} \left(\frac{A_{j_l}+1}{a-1} - \frac{A_{j_{l+1}}}{a} \right). \end{aligned} \tag{9}$$

For a lower bound, we note that if

$$a - 1 > A_{j_1} > A_{j_2} > \ldots > A_{j_k} > 0,$$

then surely $x_{j_1} > \ldots > x_{j_k}$. Hence, we can establish the following lower bound:

$$\begin{aligned} P &\geq \sum_{A_{j_1}=1}^{a-1} \sum_{A_{j_2}=1}^{A_{j_1}-1} \cdots \sum_{A_{j_k}=1}^{A_{j_{k-1}}-1} \frac{I(A_1,\ldots,A_k)}{a^k} \\ &> \sum_{A_{j_1}=1}^{a-1} \sum_{A_{j_2}=1}^{A_{j_1}-1} \cdots \sum_{A_{j_k}=1}^{A_{j_{k-1}}-1} \frac{1}{a^k}\left(\frac{A_{j_l}}{a} - \frac{A_{j_{l+1}}+1}{a-1}\right). \end{aligned} \tag{10}$$

Thus, the key to estimating P is to calculate sums of the form

$$\sum_{B_1=0}^{n} \sum_{B_2=0}^{B_1} \cdots \sum_{B_d=0}^{B_{d-1}} B_{d-m+1} \tag{11}$$

and

$$\sum_{B_1=1}^{n-1} \sum_{B_2=1}^{B_1-1} \cdots \sum_{B_d=1}^{B_{d-1}-1} B_{d-m+1} \tag{12}$$

for integers m, $1 \leq m \leq d$.

The sum

$$\sum_{B_1=0}^{n} \sum_{B_2=0}^{B_1} \cdots \sum_{B_n=0}^{B_{d-1}} 1 \tag{13}$$

is the coefficient of x^n in the generating function $1/(1-x)^{d+1}$ or, in other words, it is given by $T_d(n+1)$.

Thus, the sum in (11) can be represented by the generating function

$$\frac{x}{(1-x)^{d-m+1}} \frac{d}{dx}\left(\frac{1}{(1-x)^m}\right) = \frac{mx}{(1-x)^{d+2}},$$

and so,

$$\sum_{B_1=0}^{n} \sum_{B_2=0}^{B_1} \cdots \sum_{B_d=0}^{B_{d-1}} B_{d-m+1} = m\,T_{d+1}(n) = m\binom{n+d}{d+1}.$$

Likewise,

$$\sum_{B_1=1}^{n-1} \sum_{B_2=1}^{B_1-1} \cdots \sum_{B_d=1}^{B_{d-1}-1} B_{d-m+1} = m\,T_{d+1}(n-d) = m\binom{n}{d+1}.$$

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As A_{j_l} is the l-th largest of the A_i's, and $A_{j_{l+1}}$ is the $(l+1)$-th largest of the A_i's, from (9) we have the following upper bound for (8):

$$\begin{aligned}
&\frac{k-l+1}{a^k(a-1)}\binom{a+k-1}{k+1}+\frac{1}{a^k(a-1)}\binom{a+k-1}{k}-\frac{k-l}{a^{k+1}}\binom{a+k-1}{k+1}\\
&= \frac{a+k-l}{a^{k+1}(a-1)}\binom{a+k-1}{k+1}+\frac{1}{a^k(a-1)}\binom{a+k-1}{k}\\
&= \frac{a+k-l}{a^{k+1}(a-1)}\binom{a+k-1}{k+1}+\frac{1}{a^k(a-1)}\frac{k+1}{a-1}\binom{a+k-1}{k+1}\\
&= \frac{a^2+2ak-(a-1)l-k}{a^{k+1}(a-1)^2}\binom{a+k-1}{k+1}\le\frac{a+2k}{a^k(a-1)^2}\binom{a+k-1}{k+1}.
\end{aligned}$$

Likewise, from (10) we have the following lower bound for (8):

$$\begin{aligned}
&\frac{k-l+1}{a^{k+1}}\binom{a}{k+1}-\frac{k-l}{a^k(a-1)}\binom{a}{k+1}-\frac{1}{a^k(a-1)}\binom{a-1}{k}\\
&=\frac{a-k+l-1}{a^{k+1}(a-1)}\binom{a}{k+1}-\frac{k+1}{a^{k+1}(a-1)}\binom{a}{k+1}\\
&=\frac{a-2k+l-2}{a^{k+1}(a-1)}\binom{a}{k+1}\ge\frac{a-2k-2}{a^{k+1}(a-1)}\binom{a}{k+1}.
\end{aligned}$$

Thus, we have shown that

$$\frac{a-2k-2}{a^{k+1}(a-1)}\binom{a}{k+1}<P<\frac{a+2k}{a^k(a-1)^2}\binom{a+k-1}{k+1}. \tag{14}$$

We will now show that P has uniform bounds in terms of a and k, for all $a\ge 2$ and $k\ge 2$

$$\frac{1}{(k+1)!}-\frac{3}{a}<P<\frac{1}{(k+1)!}+\frac{3}{a-1}.$$

Consider

$$\begin{aligned}
U &= \frac{1}{(k+1)!}+\frac{3}{a-1}-\frac{a+2k}{a^k(a-1)^2}\binom{a+k-1}{k+1}\\
&= \frac{a^k(a-1)+3a^k(k+1)!-(a+2k)(a+k-1)\cdots a}{a^k(a-1)(k+1)!}.
\end{aligned}$$

Note that all the coefficients in the polynomial $f(a)=(a+2k)(a+k-1)\cdots a$ are positive. By substituting $a=1$, we see that they sum up to $(1+2k)k!<2(k+1)!$. Thus,

$$U > \frac{a^k(a-1)+3a^k(k+1)!-(a^{k+1}+(2k+\sum_{i=1}^{k-1}i)a^k+2a^k(k+1)!)}{a^k(a-1)(k+1)!}$$

$$= \quad \frac{a^k(k+1)! - 0.5(k+1)(k+2)a^k}{a^k(a-1)(k+1)!} \geq 0.$$

The last inequality follows as $2(k+1)! \geq (k+1)(k+2)$, for $k \geq 2$.

Now, consider

$$\begin{aligned} L &= \frac{a-2k-2}{a^{k+1}(a-1)} \binom{a}{k+1} + \frac{3}{a} - \frac{1}{(k+1)!} \\ &= \frac{(a-2(k+1))(a-2)\cdots(a-k) + 3a^{k-1}(k+1)! - a^k}{a^k(k+1)!}. \end{aligned}$$

Consider the expansion of the polynomial $f(a) = (a-2(k+1))(a-2)\cdots(a-k)$

$$\begin{aligned} f(a) &= a^k - \left(2(k+1) + \sum_{2\leq i\leq k} i\right) a^{k-1} + \\ &\quad + \left(2(k+1)\sum_{2\leq i\leq k} i + \sum_{2\leq i<j\leq k} ij\right) a^{k-2} - \cdots + (-1)^k 2(k+1)!. \end{aligned}$$

In the expansion of the polynomial $(a-2)\cdots(a-k)$, the absolute value of the coefficient of a^r will not exceed

$$\binom{k-1}{r} k! = \frac{(k-1)!k!}{r!(k-r-1)!} \leq (k-1)!k!.$$

Since, for $a \leq (k+1)!$, L is trivially positive, we consider $a \geq (k+1)!$. In this case, we see that, in the expansion of the polynomial $f(a)$, the absolute value of the term involving a^r, for $k > 1$, will not exceed

$$(2(k+1)(k-1)!k! + (k-1)!k!)\, a^r < 2(k+1)!k!a^r < 2k!a^{r+1}.$$

Now, discarding the positive coefficients of a^r, for $r < k-1$ and bounding the $\lceil \frac{k}{2} \rceil - 1 \leq \frac{k}{2}$ negative terms by $-2k!a^{k-1}$, we have

$$f(a) > a^k - 0.5(k^2+5k+2)a^{k-1} - kk!a^{k-1}.$$

However, for $k \geq 2$, $2(k+1)! > 0.5(k^2+5k+2)$ and so, $f(a) > a^k - 3(k+1)!a^{k-1}$. Thus,

$$L > \frac{a^k - 3(k+1)!a^{k-1} + 3a^{k-1}(k+1)! - a^k}{a^k(k+1)!} = 0.$$

Thus, the desired result follows. □

3. Discussion

In Theorem 2.1 we demonstrated that equidistributed multiply sequences are not m-equipartitioned for any $m > 2$ by establishing an exact value for $P(x_i > x_{i+i} > \cdots > x_{i+m-1})$. Calculations for other individual permutations can be established in a similar fashion. For example, for $m = 3$ it is easy to show that:

$$\begin{aligned} & P(x_i > x_{i+1} > x_{i+2}) = P(x_{i+2} > x_{i+1} > x_i) = \frac{1}{6}\left(1 + \frac{1}{a}\right), \\ & P(x_{i+1} > x_i > x_{i+2}) = P(x_{i+2} > x_i > x_{i+1}) = \frac{1}{6}\left(1 - \frac{1}{a}\right), \\ \text{and} \quad & P(x_i > x_{i+2} > x_{i+1}) = P(x_{i+1} > x_{i+2} > x_i) = \frac{1}{6}. \end{aligned}$$

Exact values for $m > 3$ can likewise be established, although they become increasingly more complex.

However, in Theorem 2.2 we established a uniform bound for all equipartition tests of equidistributed multiply sequences. As multiply sequences are *almost* ∞-distributed in an asymptotic sense, these bounds can be thought of as one measure of how close a sequence is to ∞-distribution.

In comparing the general bounds of Theorem 2.2 to the above permutation tests for $m = 3$ it is apparent that the bounds can be improved. We have provided some alternate bounds, such as inequality (14), in intermediary steps. For large a and m, other bounds might be possible from our calculations by using Stirling-like approximations [5].

We conjecture in fact that Theorem 2.1 provides the worst case upper bound, and a symmetric worst case lower bound, as follows:

Conjecture 1. *Let $x_n = ax_{n-1}$ mod 1 be an equidistributed sequence, with $a > 1$ an integer and $x_0 \in [0, 1)$. Then, for any permutation $i_0, \ldots, i_k$ of the index set $\{i, \ldots, i + k\}$, we have*

$$\left| P(x_{i_0} > x_{i_2} > \cdots > x_{i_k}) - \frac{1}{(k+1)!} \right| \le \frac{1}{a^k(a-1)} \binom{a+k-1}{k+1} - \frac{1}{(k+1)!}.$$

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