Math 1100B — Calculus, Test #2 — 2009-02-03

(10) 1. Find $\lim_{x\to\infty} x^2 e^{-x}$.

Solution: Since

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x}$$

and $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} e^x = \infty$, l'Hospital's Rule applies:

$$\begin{aligned}\lim_{x\to\infty} x^2 e^{-x} &= \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} \\ &= \lim_{x\to\infty} \frac{2}{e^x} = \boxed{0}.\end{aligned}$$

□

(15) 2. You operate an ice cream factory. The market price of ice cream is \$50 per barrel, and at this price, you can sell as much ice cream as you want to produce. The cost of producing x barrels of ice cream per day is $f(x) = x^3 + \frac{9}{2}x^2 + 20x + 1$. What quantity of ice cream should you produce per day, so as to maximize your profits?

Recall: [profit] = [revenue]-[cost], and [revenue] = [price] x [quantity sold].

Solution: Selling x barrels of ice cream yields $50x$ dollars of revenue, but incurs $f(x)$ dollars of cost, for a net profit of $p(x) = 50x - f(x) = -x^3 - \frac{9}{2}x^2 + 30x - 1$. Thus, $p'(x) = -3x^2 - 9x + 30 = -3(x^2 + 3x - 10) = -3(x+5)(x-2)$, which has roots $x = 2$ and $x = -5$. Clearly, a 'negative' quantity of ice cream is nonsensical, so we must have $x = 2$. To verify that this quantity *maximizes* (not *minimizes*) profits, we observe that $p''(x) = -6x - 9$ so that $p''(2) = -21 < 0$; thus, the function p is maximal at this point.

Thus, producing $\boxed{\text{2 barrels}}$ of ice-cream per day is profit-maximizing. □

(15) 3. Suppose $g(x) := \int_{\sqrt{x}}^{x} e^{t^2}\,dt$ for all x. Find $g'(x)$.

Solution:

$$\begin{aligned}\frac{d}{dx}\left(\int_{\sqrt{x}}^{x} e^{t^2}\,dt\right) &= \frac{d}{dx}\left(\int_{\sqrt{x}}^{1} e^{t^2}\,dt + \int_{1}^{x} e^{t^2}\,dt\right) \\ &= \frac{d}{dx}\left(-\int_{1}^{\sqrt{x}} e^{t^2}\,dt + \int_{1}^{x} e^{t^2}\,dt\right) \\ &= \boxed{-e^{x}\left(\frac{1}{2\sqrt{x}}\right) + e^{x^2}}.\end{aligned}$$

□

4. Compute the following integrals:

(15) (a) $\displaystyle\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\ dx.$

Solution: $\displaystyle\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\ dx \underset{(s)}{=} 2\int \cos(u)\ du \underset{(*)}{=} 2\sin(u) + C \underset{(s)}{=} \boxed{2\sin(\sqrt{x}) + C.}$

Here, (s) is the substitution $u := \sqrt{x}$, so that $du = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}\ dx$. Equality $(*)$ is because $\sin'(x) = \cos(x)$. □

(15) (b) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\ dx.$

Solution: $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\ dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\ du \underset{(*)}{=} \frac{1}{2}\arcsin(u) + C \underset{(s)}{=} \boxed{\frac{1}{2}\arcsin(x^2) + C}.$

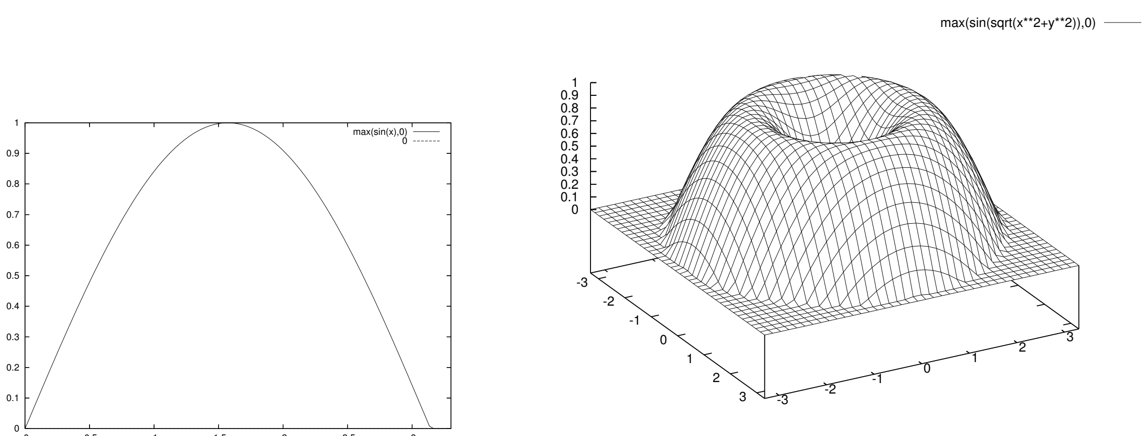
Here, (s) is the substition $u := x^2$, so that $du = 2x\ dx$. Equality $(*)$ is because $\arcsin'(u) = \frac{1}{\sqrt{1-u^2}}$. □

(15) (c) $\displaystyle\int (\ln(x))^2\ dx.$

Solution:

$$\begin{aligned}
\int \ln(x)^2\ dx &\underset{(p)}{=} \ln(x)^2 x - \int \frac{2\ln(x)}{x}\cdot x\ dx = \ln(x)^2 x - 2\int \ln(x)\ dx \\
&\underset{(\P)}{=} \ln(x)^2 x - 2\ln(x)x + 2\int \frac{1}{x}x\ dx = \ln(x)^2 x - 2\ln(x)x + 2\int 1\ dx \\
&= \boxed{\ln(x)^2 x - 2\ln(x)x + 2x + C.}
\end{aligned}$$

Here (p) is integration by parts, with $u = \ln(x)^2$ and $dv = dx$, so that $du = 2\ln(x)/x$ and $v = x$. Then (¶) is integration by parts again, this time with $u := \ln(x)$ and $dv = dx$, so that $du = 1/x$ and $v = x$ (or, for this part you can just refer to Example 2 on page 454 in the book, where we computed $\int \ln(x)\ dx$). □



5. Let $\mathcal{R}$ be the region in the plane bounded by the x axis and the curve $y = \sin(x)$ between
(15) $x = 0$ and $x = \pi$ (see left side of figure). Find the volume of the solid obtained by rotating

$\mathcal{R}$ around the y axis (see right side of figure).

Solution: We use the 'Method of Cylindrical Shells'. We have

$$
\begin{aligned}
V &= 2\pi \int_0^\pi x \sin(x)\ dx \underset{(*)}{=} 2\pi \left(-x\cos(x) \Big|_{x=0}^{x=\pi} + \int_0^\pi \cos(x)\ dx \right) \\
&= 2\pi \left(-\pi \underbrace{\cos(\pi)}_{=-1} + 0\ \cos(0) + \sin(x) \Big|_{x=0}^{x=\pi} \right) \\
&= 2\pi \left(\pi + \pi \underbrace{\sin(\pi)}_{=0} - 0\ \sin(0) \right) = \boxed{2\pi^2.}
\end{aligned}
$$

Here $(*)$ is integration by parts, with $u := x$, and $dv := \sin(x)$, so that $du := dx$ and $v = -\cos(x)$.
$\square$