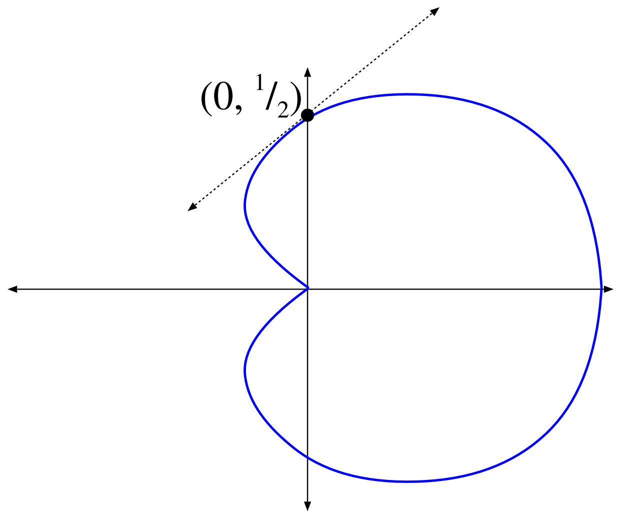
Math 1100B — Calculus, Test #1 — 2008-11-04



(20) 1. The set of solutions to the equation

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$

forms a curve called a *cardioid*. Use implicit differentiation to find the slope of the cardioid at the point $(x, y) = (0, \frac{1}{2})$.

Solution: Suppose y is a function of x —that is $y = y(x)$. Differentiating both sides of the cardioid equation, we get

$$2x + 2y(x)y'(x) = 2(2x^2 + 2y(x)^2 - x) \cdot (4x + 4y(x)y'(x) - 1), \tag{1}$$

Substituting $(x, y) = (0, \frac{1}{2})$, we get

$$0 + 2\left(\frac{1}{2}\right) y'(x) = 2\left(0 + 2\left(\frac{1}{2}\right)^2 - 0\right) \cdot \left(0 + 4\left(\frac{1}{2}\right) y'(x) - 1\right),$$

which simplifies to

$$y'(x) = 2y'(x) - 1,$$

which simplifies to

$$1 = y'(x).$$

Thus, the slope is equal to $\boxed{1.}$

Although it is not necessary in order to answer this question, one can also obtain an expression for $y'(x)$ as a function of x and $y(x)$, by simplifying eqn. (1) as follows:

$$\begin{aligned} x + y(x)y'(x) &= (2x^2 + 2y(x)^2 - x) \cdot (4x + 4y(x)y'(x) - 1), \\ &= 4(2x^2 + 2y(x)^2 - x)\, y(x)y'(x) + (2x^2 + 2y(x)^2 - x) \cdot (4x - 1). \end{aligned}$$

Thus,

$$\begin{aligned} x-(2x^2+2y(x)^2-x)\cdot(4x-1) &= 4(2x^2+2y(x)^2-x-1)\,y(x)y'(x); \\ \text{Thus, } \frac{x-(2x^2+2y(x)^2-x)\cdot(4x-1)}{4(2x^2+2y(x)^2-x-1)\,y(x)} &= y'(x); \end{aligned}$$

This simplifies to

$$y'(x) = \boxed{2\,\frac{4xy^2+4x^3-y^2-3x^2}{y+4xy-8x^2y-8y^3}.}$$

□

(25) 2. Let $f(x)=\sqrt{2+5x}$. Use the 'limit' definition of the derivative to show that

$$f'(x) = \frac{5}{2\sqrt{2+5x}}.$$

Solution:

$$\begin{aligned} f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{\sqrt{2+5x}-\sqrt{2+5a}}{x-a} \\ &= \lim_{x\to a}\left(\frac{\sqrt{2+5x}-\sqrt{2+5a}}{x-a}\right)\left(\frac{\sqrt{2+5x}+\sqrt{2+5a}}{\sqrt{2+5x}+\sqrt{2+5a}}\right) \\ &= \lim_{x\to a}\frac{(\sqrt{2+5x})^2-(\sqrt{2+5a})^2}{(x-a)\left(\sqrt{2+5x}+\sqrt{2+5a}\right)} = \lim_{x\to a}\frac{2+5x-(2+5a)}{(x-a)\left(\sqrt{2+5x}+\sqrt{2+5a}\right)} \\ &= \lim_{x\to a}\frac{5(x-a)}{(x-a)\left(\sqrt{2+5x}+\sqrt{2+5a}\right)} = \lim_{x\to a}\frac{5}{\sqrt{2+5x}+\sqrt{2+5a}} \\ &= \frac{5}{\sqrt{2+5a}+\sqrt{2+5a}} = \boxed{\frac{5}{2\sqrt{2+5a}}} \end{aligned}$$

□

(25) 3. Find the horizontal and vertical asymptotes (if they exist) of the curve

$$f(x) = \frac{5\sin(3x)}{x\,(x-2)}.$$

Solution: Since

$$0 \le \left|\frac{5\sin(3x)}{x\,(x-2)}\right| \le \left|\frac{5}{x\,(x-2)}\right|$$

and

$$\lim_{x\to\infty}\left|\frac{5}{x\,(x-2)}\right| = 0,$$

we have

$$\lim_{x\to\infty}\left|\frac{5\sin(3x)}{x\,(x-2)}\right| = 0.$$

Therefore

$$\lim_{x\to\infty} \frac{5\sin(3x)}{x\,(x-2)} = 0.$$

Similarly, we have

$$\lim_{x\to-\infty} \frac{5\sin(3x)}{x\,(x-1)} = 0.$$

Therefore, $\boxed{y = 0 \text{ is the horizontal asymptote.}}$

Now,

$$\lim_{x\to 0} \frac{5\sin(3x)}{x\,(x-2)} = \lim_{x\to 0} \frac{5\times 3}{(x-2)}\frac{\sin(3x)}{3x} = -\frac{15}{2},$$

Thus, $\boxed{x = 0 \text{ is not a vertical asymptote.}}$ We note that $\pi < 6 < 2\pi$, so $\sin 6 < 0$ and

$$\lim_{x\to 2^-} \frac{5\sin(3x)}{x\,(x-2)} = \infty,$$

$$\lim_{x\to 2^+} \frac{5\sin(3x)}{x\,(x-2)} = -\infty.$$

Therefore $\boxed{x = 2 \text{ is the vertical asymptote.}}$ □

4. Calculate $f'(x)$ for the following two functions.

(15) (a) $f(x) := \dfrac{\sin(x)\cos(x)}{\ln(x)}$.

Solution:

$$\begin{aligned} f'(x) &= \frac{(\cos(x)\cos(x) - \sin(x)\sin(x))\ln(x) - \sin(x)\cos(x)\cdot\frac{1}{x}}{\ln^2(x)} \\ &= \boxed{\frac{(\cos(x)^2 - \sin(x)^2)\ln(x) - \sin(x)\cos(x)/x}{\ln(x)^2}.} \end{aligned}$$

□

(15) (b) $f(x) := e^{\sqrt{1+x^2}}$.

Solution:

$$f'(x) = \exp(\sqrt{1+x^2})\cdot\frac{1}{2\sqrt{1+x^2}}\cdot 2x = \boxed{\frac{x\,\exp(\sqrt{1+x^2})}{\sqrt{1+x^2}}.}$$

□