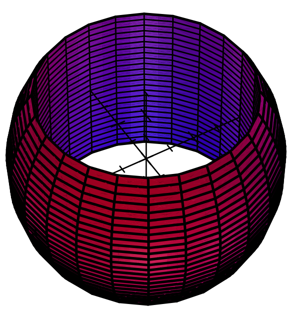
MATH 1100 — Homework #3 — Due Thursday, February 26, 2009

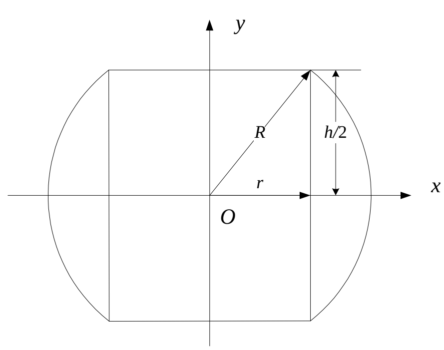
1. (5 points) p. 437. #46. Suppose you make napkin rings by drilling holes with different diameters through two wooden balls (which also have different diameters). You discover that both napkin rings have the same height h, as shown in the figure.

   (a) Guess which ring has more wood in it.

   (b) Check your guess: Use cylindrical shells to compute the volume of a napkin ring created by drilling a hole with radius r through the center of a sphere of radius R and express the answer in terms of h.

   Solution:





Since the napkin ring is symmetric, we only need to calculate the volume of the upper half of the ring. This solid is obtained by rotating the region bounded by the curves $y = \sqrt{R^2 - x^2}$, $x = r$ and $y = 0$ about the y axis. Using the shell method, we have

$$V = \int_r^R 2\pi x \sqrt{R^2 - x^2} dx$$

$$= -\pi \int_{R^2-r^2}^{0} \sqrt{u} du$$

where $u = R^2 - x^2$, $du = -2x dx$,

$$= -\pi \left[\frac{2}{3} u^{\frac{3}{2}} \right]_{R^2-r^2}^{0} = \frac{2\pi}{3} \left(R^2 - r^2 \right)^{\frac{3}{2}}.$$

Since $R^2 - r^2 = \left(\frac{h}{2}\right)^2$, we have $V = \frac{2\pi}{3} \left(\frac{h}{2}\right)^3$. The volume of the napkin ring is $2V = \frac{\pi h^3}{6}$. The volume only depends on h. Therefore, any two rings with the same height have the same amount of wood.

2. Evaluate the integral. (2.5 points each)

   (a) $\int \frac{dx}{\sqrt{e^x - 1}}$

   Solution: Let $u = e^x - 1$. $x = \ln(u+1)$. $dx = \frac{du}{u+1}$.

$$\int \frac{dx}{\sqrt{e^x - 1}}$$

$$= \int \frac{du}{(u+1)\sqrt{u}}$$

Let $v = \sqrt{u}$. $u = v^2$. $du = 2v dv$.

$$= \int \frac{2v dv}{(v^2+1) v} = 2 \int \frac{dv}{1 + v^2}$$

$$= 2 \arctan(v) + C = 2 \arctan\left(\sqrt{u}\right) + C$$

$$= 2 \arctan\left(\sqrt{e^x - 1}\right) + C.$$

   (b) $\int \frac{1}{\sqrt{4y^2 - 4y - 3}} dy$

   Solution: Since

$$4y^2 - 4y - 3 = 4y^2 - 4y + 1 - 4 = (2y-1)^2 - 4,$$

we let $2y - 1 = 2\sec\theta$. $y = \frac{2\sec\theta + 1}{2}$. $dy = \sec\theta \tan\theta d\theta$.

$$\int \frac{1}{\sqrt{4y^2 - 4y - 3}} dy$$

$$= \int \frac{dy}{\sqrt{(2y-1)^2 - 4}} = \int \frac{\sec\theta \tan\theta d\theta}{\sqrt{4\sec^2\theta - 4}}$$

$$= \int \frac{\sec\theta \tan\theta d\theta}{2\tan\theta} = \frac{1}{2} \int \sec\theta d\theta$$

$$= \frac{1}{2} \ln|\sec\theta + \tan\theta| + C$$

We have $\sec\theta = \frac{2y-1}{2}$ and $\tan\theta = \sqrt{\left(\frac{2y-1}{2}\right)^2 - 1} = \frac{\sqrt{4y^2-4y-3}}{2}$.

$$\frac{1}{2} \ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{2} \ln\left| \frac{2y-1}{2} + \frac{\sqrt{4y^2 - 4y - 3}}{2} \right| + C$$

$$= \frac{1}{2} \ln\left| 2y - 1 + \sqrt{4y^2 - 4y - 3} \right| + C'.$$

(c) $\int \frac{x^2+8x-3}{x^3+3x^2}dx$

Solution: Let

$$\frac{x^2+8x-3}{x^3+3x^2} = \frac{x^2+8x-3}{x^2(x+3)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+3}$$

$x^2+8x-3 = Ax(x+3)+B(x+3)+Cx^2$

Let $x=0$. We have

$$-3 = 3B \Rightarrow B = -1.$$

Let $x=-3$. We have

$$9-24-3 = -18 = C(9) \Rightarrow C = -2.$$

Let $x=1$. We have

$$1+8-3 = 6 = A(4)-4-2 \Rightarrow A = 3.$$

Therefore

$$\frac{x^2+8x-3}{x^3+3x^2} = \frac{3}{x}-\frac{1}{x^2}-\frac{2}{x+3}$$

$$\int \frac{x^2+8x-3}{x^3+3x^2}dx = \int \left(\frac{3}{x}-\frac{1}{x^2}-\frac{2}{x+3}\right)dx$$

$$= 3\ln x+\frac{1}{x}-2\ln|x+3|+C.$$

(d) $\int te^{\sqrt{t}}dt$

Solution: Let $u=\sqrt{t}$. $t=u^2$. $dt=2udu$.

$$\int te^{\sqrt{t}}dt$$

$$= \int u^2e^u(2u)\,du = 2\int u^3e^u du$$

$$= 2\left(u^3e^u-\int e^u 3u^2du\right) = 2u^3e^u-6\int u^2e^u du$$

$$= 2u^3e^u-6\left(u^2e^u-\int e^u 2udu\right)$$

$$= 2u^3e^u-6u^2e^u+12\int ue^u du$$

$$= 2u^3e^u-6u^2e^u+12\left(ue^u-\int e^u du\right)$$

$$= 2u^3e^u-6u^2e^u+12ue^u-12e^u+C$$

$$= 2t^{\frac{3}{2}}e^{\sqrt{t}}-6te^{\sqrt{t}}+12\sqrt{t}e^{\sqrt{t}}-12e^{\sqrt{t}}+C.$$

(e) $\int \frac{dx}{\sqrt{x}+x\sqrt{x}}$

Solution: Let $u=\sqrt{x}$. $x=u^2$. $dx=2udu$.

$$\int \frac{dx}{\sqrt{x}+x\sqrt{x}}$$

$$= \int \frac{2udu}{u+u^3} = 2\int \frac{du}{1+u^2}$$

$$= 2\arctan(u)+C$$

$$= 2\arctan\left(\sqrt{x}\right)+C.$$

(f) $\int \frac{1+\sin x}{1-\sin x}dx$

Solution:

$$\int \frac{1+\sin x}{1-\sin x}dx$$

$$= \int \frac{(1+\sin x)(1+\sin x)}{(1-\sin x)(1+\sin x)}dx$$

$$= \int \frac{1+2\sin x+\sin^2 x}{1-\sin^2 x}dx$$

$$= \int \frac{1+2\sin x+\sin^2 x}{\cos^2 x}dx$$

$$= \int \frac{1}{\cos^2 x}dx+\int \frac{2\sin x}{\cos^2 x}dx+\int \frac{\sin^2 x}{\cos^2 x}dx$$

Since

$$\int \frac{1}{\cos^2 x}dx = \int \sec^2 x dx = \tan x + C$$

and let $u=\cos x$, $du=-\sin x dx$,

$$\int \frac{2\sin x}{\cos^2 x}dx$$

$$= -2\int \frac{-\sin x dx}{\cos^2 x} = -2\int \frac{du}{u^2}$$

$$= \frac{2}{u}+C = \frac{2}{\cos x}+C$$

and

$$\int \frac{\sin^2 x}{\cos^2 x}dx$$

$$= \int \tan^2 x dx = \int \left(\sec^2 x-1\right)dx$$

$$= \int \sec^2 x dx - \int dx = \tan x - x + C.$$

Therefore,

$$\int \frac{1+\sin x}{1-\sin x}dx$$

$$= \tan x+\frac{2}{\cos x}+\tan x-x+C$$

$$= 2\tan x+\frac{2}{\cos x}-x+C.$$