Math 1100 — Homework #2 — Due Thursday, December ??, 2008

Calculus of rainbows (p.279).
`(4 pts)`
#1 Snell's law says $\sin(\alpha) = k\,\sin(\beta)$, which means that $\beta = \arcsin(\sin(\alpha)/k)$. Thus, the angle of deviation $D(\alpha)$ is given by

$$D(\alpha) \quad = \quad \pi + 2\alpha - 4\beta \quad = \quad \pi + 2\alpha - 4\arcsin(\sin(\alpha)/k).$$

Thus,

$$\begin{aligned} D'(\alpha) &= 2 - 4\arcsin'(\sin(\alpha)/k)\cdot\sin'(\alpha)/k \quad = \quad 2 - \frac{4\cos(\alpha)/k}{\sqrt{1-\sin(\alpha)^2/k^2}} \\ &= 2 - \frac{4\cos(\alpha)}{\sqrt{k^2-\sin(\alpha)^2}}. \end{aligned}$$

Thus $D'(\alpha) = 0$ if and only if $2 - \frac{4\cos(\alpha)}{\sqrt{k^2-\sin(\alpha)^2}} = 0$,

$$\begin{array}{lrcl} \text{which is equivalent to} & 4\cos(\alpha) & = & 2\sqrt{k^2-\sin(\alpha)^2}, \\ \text{which is equivalent to} & 4\cos(\alpha)^2 & = & k^2-\sin(\alpha)^2, \\ \text{which is equivalent to} & 4\cos(\alpha)^2+\sin(\alpha)^2 & = & k^2, \\ \text{which is equivalent to} & 3\cos(\alpha)^2+1 & = & k^2, \\ \text{which is equivalent to} & 3\cos(\alpha)^2 & = & k^2-1, \\ \text{which is equivalent to} & \cos(\alpha) & = & \sqrt{\frac{k^2-1}{3}}, \\ \text{which is equivalent to} & \alpha & = & \arccos\left(\sqrt{\frac{k^2-1}{3}}\right). \end{array}$$

Thus, if $k = \frac{4}{3}$, then we get

$$\begin{aligned} \alpha &= \arccos\left(\sqrt{\frac{(4/3)^2-1}{3}}\right) \quad = \quad \arccos\left(\sqrt{\frac{(16/9)-(9/9)}{3}}\right) \\ &= \arccos\left(\sqrt{\frac{7}{27}}\right) \quad \approx \quad 59.3911^o. \end{aligned} \tag{1}$$

And thus,

$$\begin{aligned} D(\alpha) &= \pi + 2\alpha - 4\arcsin(\sin(\alpha)/k) \underset{(*)}{=} 180^o + 2(59.3911^o) - 4\arcsin(3\sin(59.3911^o)/4) \\ &= 180^o + 118.8 - 160.83 \quad = \quad 137.9^o. \end{aligned} \tag{2}$$

where $(*)$ involves conversion from radian units to degrees.

(2 pts)

#2. Repeating computations (1) and (2) for $k_{red} = 1.3318$ and $k_{vio} = 1.3435$, we get

$$\begin{aligned} \alpha_{red} &= 59.4802^o \\ \text{and } \alpha_{vio} &= 58.8013^o, \end{aligned}$$

and thus,

$$\begin{aligned} D(\alpha_{red}) &= 42.3^o \\ \text{and } D(\alpha_{vio}) &= 40.6^o. \end{aligned}$$

(4 pts)

 # 17. on p.352 **Theorem.** *Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ be constants, and let $f(x) = a_1 \sin(x) + a_2 \sin(2x) + a_3 \sin(3x) + \cdots + a_n \sin(nx)$. Suppose $|f(x)| \leq |\sin(x)|$ for all $x \in \mathbb{R}$. Then $|a_1 + 2a_2 + 3a_3 + \cdots + na_n| \leq 1$.*

Proof. First observe that

$$f'(x) = a_1 \cos(x) + 2a_2 \cos(2x) + 3a_3 \cos(3x) + \cdots + na_n \cos(nx)$$

and thus,

$$\begin{aligned} |f'(0)| &= |a_1 \cos(0) + 2a_2 \cos(0) + 3a_3 \cos(0) + \cdots + na_n \cos(0)| \\ &= |a_1 + 2a_2 + 3a_3 + \cdots + na_n|. \end{aligned} \tag{3}$$

This suggests that the conclusion of the theorem is somehow a statement about $f'(0)$. But

$$\begin{aligned} f'(0) &\underset{\text{(D)}}{=\!=} \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \underset{(*)}{=\!=} \lim_{x\to 0} \frac{f(x)}{x} \\ &\underset{(\dagger)}{=\!=} \lim_{x\to 0} \left(\frac{f(x)}{\sin(x)}\right) \cdot \left(\frac{\sin(x)}{x}\right) = \left(\lim_{x\to 0} \frac{f(x)}{\sin(x)}\right) \cdot \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right) \\ &\underset{(\ddagger)}{=\!=} \lim_{x\to 0} \frac{f(x)}{\sin(x)}. \end{aligned}$$

Here, (D) is just the definition of the derivative, $(*)$ is because $f(x) = 0$, $(\dagger)$ is just multiplying-and-dividing by $\sin(x)$, and $(\ddagger)$ is because we know $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Thus,

$$|f'(0)| = \left|\lim_{x\to 0} \frac{f(x)}{\sin(x)}\right| = \lim_{x\to 0} \frac{|f(x)|}{|\sin(x)|} \underset{(*)}{\leq} 1, \tag{4}$$

where $(*)$ is because $|f(x)| \leq |\sin(x)|$ for all $x \in \mathbb{R}$, by hypothesis.

Combining equations (3) and (4) we conclude that $|a_1 + 2a_2 + 3a_3 + \cdots + na_n| \leq 1$, as desired. □

Remark. The functions like $f(x) = a_1 \sin(x) + a_2 \sin(2x) + a_3 \sin(3x) + \cdots + a_n \sin(nx)$ are called *trigonometric polynomials*. The study of such polynomials is part of *Fourier analysis*, an area of mathematics with important applications in probability theory, signal analysis, and the solution of differential equations.