Math 1100 — Homework #1 — Due Thursday, October 30, 2008

#1. A **(one-dimensional) dynamical system** is a function $f : [0,1] \longrightarrow [0,1]$. This represents a system evolving over time, whose 'state' at each moment in time can be described by a single number $x \in [0,1]$. If the state of the system at time zero is x_0, then its state at time 1 is $x_1 = f(x_0)$, its state at time 2 is $x_2 = f(x_1)$, its state at time 3 is $x_3 = f(x_2)$, and so on.

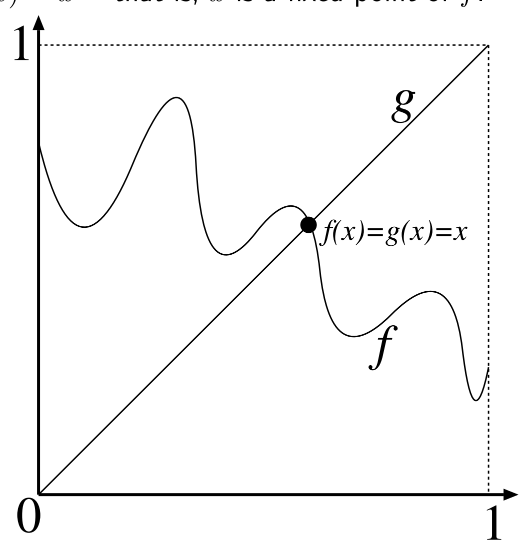
The state x is called a **fixed point** if $f(x) = x$. This means that if the state at time zero is x, then the state at time 1 is also x, and the state at time 2 is still x, and so on. Thus, fixed points represents 'equilibria' of the system; they are very important in the analysis of dynamical systems.

(a) Let $g : [0,1] \longrightarrow [0,1]$ be the function $g(x) = x$. Draw the graph of g.

Let $f : [0,1] \longrightarrow [0,1]$ be some continuous function whose range is all of $[0,1]$. Sketch a graph for f. Notice that it seems impossible to draw a graph for f that *doesn't* intersect
(10) g somewhere. Prove the following statement:

> *If (x, y) is an intersection of the graph of f with the graph of g, then x is a fixed point of f.*

Solution: An intersection point between f and g is a place where $f(x) = g(x)$. But $g(x) = x$, so this is the same as saying $f(x) = x$ —that is, x is a fixed point of f.

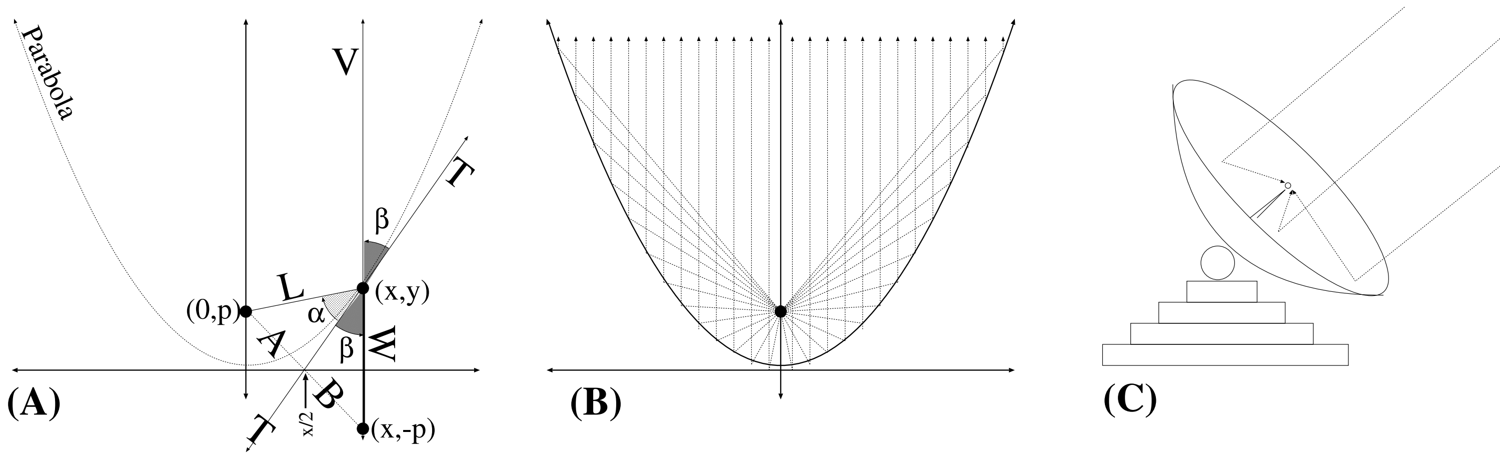


□

(20) **(b)** Use the Intermediate Value Theorem to prove the following statement:

> *If $f : [0,1] \longrightarrow [0,1]$ is any continuous function whose range is all of $[0,1]$, then f has at least one fixed point in $[0,1]$.*

Solution: Consider the function $h(x) = f(x) - x$. We must show that there is some $x \in [0,1]$ such that $f(x) = x$. But $f(x) = x$ if and only if $h(x) = 0$. Thus, it is equivalent to show that there is some $x \in [0,1]$ such that $h(x) = 0$.



The range of f is $[0,1]$. Thus, there is some $x_0 \in [0,1]$ such that $f(x_0) = 0$, which means that $h(x_0) = f(x_0) - x_0 = 0 - x_0 \leq 0$ (because $x_0 \geq 0$).

Also, there is some $x_1 \in [0,1]$ such that $f(x_1) = 1$, which means that $h(x_1) = f(x_1) - x_1 = 1 - x_0 \geq 0$ (because $x_1 \leq 1$).

But the function h is continuous (because f is continuous). Thus, if $h(x_0) \leq 0$ and $h(x_1) \geq 0$, then the Intermediate Value Theorem says there is some x between x_0 and x_1 such that $h(x) = 0$; this means that $f(x) = x$, which means that x is a fixed point of f.

Remark. You have proved the 'one-dimensional' version of a very important theorem called *Brouwer's Fixed Point Theorem*. □

#2. Let $p > 0$, and define the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = x^2/4p$. The graph of f is called a *parabola*; it is the set of all (x, y) such that $y = x^2/4p$. The *focus* of this parabola is the point $(0, p)$ on the y axis.

Let (x, y) be a point on the parabola. As shown in Figure (A), let $\mathbf{T}$ be the tangent line to f at (x, y). Let $\mathbf{L}$ be the line segment from $(0, p)$ to (x, y), and let $\mathbf{V}$ be the *vertical* line segment going upwards from (x, y) (that is, the set of all points (x, y'), for all $y' \geq y$). We will prove the following:

Theorem. *For any (x, y) on the parabola, the angle α between $\mathbf{T}$ and $\mathbf{L}$ is equal to the angle β between $\mathbf{V}$ and $\mathbf{L}$.*

(15) **(a)** Compute the slope of $\mathbf{T}$. Show that $\mathbf{T}$ intersects the x axis at the point $(x/2, 0)$.

Solution: Since (x, y) is a point on the parabola, we have $y = x^2/4p$. The slope of $\mathbf{T}$ is the derivative of f at x; that is, $f'(x) = 2x/4p = x/2p$. Observe that $(x/2p) \cdot (-x/2) = -x^2/4p$. In other words, the tangent line translates a horizontal displacement of $-x/2$ into a vertical displacement of $-x^2/4p$. Thus, when we move horizontally from x to $x - (x/2) = x/2$, we move vertically from $y = x^2/4p$ to $y = 0$ —that is, we intersect the x axis. □

(b) Let $\mathbf{W}$ be the vertical line segment from (x, y) to the point $(x, -p)$ shown Figure (A).
(10) Show that the length of $\mathbf{W}$ is equal to the length of $\mathbf{L}$.

Solution: By Pythagoras law,

$$\begin{aligned}
\text{length}(\mathbf{L}) &= \sqrt{(x-0)^2+(y-p)^2} &=& \sqrt{x^2+\left(\frac{x^2}{4p}-p\right)^2}; \\
\text{thus,}\quad \text{length}(\mathbf{L})^2 &= x^2+\left(\frac{x^2}{4p}-p\right)^2 &=& x^2+\frac{x^4}{16p^2}-\frac{x^2}{2}+p^2 \\
&= \frac{x^4}{16p^2}+\frac{x^2}{2}+p^2. \\
\text{Meanwhile,}\quad \text{length}(\mathbf{W}) &= y-(-p) \quad = \quad y+p &=& \frac{x^2}{4p}+p. \\
\text{thus,}\quad \text{length}(\mathbf{W})^2 &= \left(\frac{x^2}{4p}+p\right)^2 &=& \frac{x^4}{16p^2}+\frac{x^2}{2}+p^2.
\end{aligned}$$

Thus, $\text{length}(\mathbf{L})^2 = \text{length}(\mathbf{W})^2$; thus $\text{length}(\mathbf{L}) = \text{length}(\mathbf{W})$. □

(5) **(c)** Let $\mathbf{A}$ be the line segment from $(x/2, 0)$ to $(0, p)$. Let $\mathbf{B}$ be the line segment from $(x/2, 0)$ to $(x, -p)$. Show that $\mathbf{A}$ and $\mathbf{B}$ are the same length.

Solution: Clearly, $\dfrac{(x,-p)+(0,p)}{2} = (x/2,0)$. Thus, $(x/2, 0)$ is the midpoint of the line from $(x, -p)$ to $(0, p)$. Thus, $\text{length}(\mathbf{A}) = \text{length}(\mathbf{B})$. □

(5) **(d)** Deduce that the triangle with sides $\mathbf{A}, \mathbf{L}, \mathbf{T}$ and the triangle with sides $\mathbf{B}, \mathbf{W}, \mathbf{T}$ are mirror images of each other across the line $\mathbf{T}$.

Solution: Parts (b) and (c) show that $\text{length}(\mathbf{A}) = \text{length}(\mathbf{B})$ and $\text{length}(\mathbf{W}) = \text{length}(\mathbf{L})$. Thus, the two triangles are reflections across $\mathbf{T}$. □

(5) **(e)** Conclude that $\alpha = \beta$.

Solution: The reflection in part (d) carries angle α to angle β. Thus, they must be equal. □

The significance of this Theorem is that all light beams originating at $(0, p)$ and striking the parabola will be reflected into parallel vertical light beams (Figure (B)). To harness this phenomenon in three dimensions, we consider the *paraboloid*: the surface obtained by rotating the parabola 360^o about its vertical axis. A mirror in the shape of a paraboloid is called a *parabolic reflector*; these are used in flashlights, car headlights, reflection telescopes, radio telescopes, radar dishes, satelite dishes, and parabolic microphones (Figure (C)).

#3. Consider a power supply providing an electric voltage which fluctuates over time. The average voltage is $25\,\mu V$ (here μV=microVolts). The *variance* of the power supply is the maximum deviation from this average over time. For example, if the voltage fluctuates between $22.1\,\mu V$ and $26.5\,\mu V$, then its variance is $\max\{|-2.9|, |1.5|\} = 2.9\,\mu V$.

Sensitive electronic components require a very stable power supply. Thus, we pass the power supply through an *attenuator* which diminishes the fluctuations in voltage. Suppose the attenuator is described by the function $f(x) = \sqrt{x}$ —that is, if the input voltage is x, then the output voltage is $\sqrt{x}$. Thus, if the input fluctuates between $16\,\mu V$ and $36\,\mu V$, then the output will fluctuate between $4\,\mu V$ and $6\,\mu V$.

(a) Let $\epsilon > 0$, and suppose we want the average output voltage to be $5\,\mu V$, with a maximum
(25) variance of $\epsilon\,\mu V$. Clearly, the average input voltage should be $25\,\mu V$. Find an input variance $\delta > 0$ which guarantees an output variance of at most ϵ. (That is, find δ such that, if the input is between $(25-\delta)\,\mu V$ and $(25+\delta)\,\mu V$, then the output is guaranteed to be between $(5-\epsilon)\,\mu V$ and $(5+\epsilon)\,\mu V$).

Solution: We want to find some $\delta > 0$ such that, if $|x-25| < \delta$, then $|\sqrt{x}-5| < \epsilon$. Observe that

$$\sqrt{x}-5 \quad = \quad \frac{\sqrt{x}^2-5^2}{\sqrt{x}+5} \quad = \quad \frac{x-25}{\sqrt{x}+5}$$

Thus,

$$\left|\sqrt{x}-5\right| \quad \leq \quad \frac{|x-25|}{\sqrt{x}+5} \tag{1}$$

Suppose $\delta \leq 9$, and $|x-25| < \delta$. Then $16 < x < 34$. But if $x > 16$, then $\sqrt{x} > 4$. Thus, $\sqrt{x}+5 > 9$. Thus,

$$\frac{|x-25|}{\sqrt{x}+5} \quad \underset{(*)}{<} \quad \frac{|x-25|}{9} \quad \underset{(\dagger)}{<} \quad \frac{\delta}{9} \tag{2}$$

Here, $(*)$ is because $\sqrt{x}+5 > 9$, and $(\dagger)$ is because $|x-25| < \delta$. If $\delta < 9\epsilon$, then $\delta/9 < \epsilon$.

So, let $\delta = \min\{9\epsilon, 9\}$. If $|x-25| < \delta$, then

$$\left|\sqrt{x}-5\right| \quad \underset{(*)}{<} \quad \frac{\delta}{9} \quad \underset{(\dagger)}{\leq} \quad \frac{9\epsilon}{9} \quad = \quad \epsilon,$$

so that $|\sqrt{x}-5| < \epsilon$, as desired. Here, $(*)$ is by equations (1) and (2) (because $\delta \leq 9$), and $(\dagger)$ is because $\delta < 9\epsilon$. $\square$

(b) What property of the function $f(x) = \sqrt{x}$ allows you to guarantee that you can find such
(5) an input variance $\delta > 0$ for any specified output variance $\epsilon > 0$?

Solution: We are using the fact that the function $f(x) = \sqrt{x}$ is **continuous** at $x = 25$. $\square$