MATH 1100 2008 Section A Midterm Test 2

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Name ______________________________

Instructions. *All answers should be clear and complete. Show all your work.* (Total 2 pages and 20 points.)

1. (9 points) Evaluate (Do not simplify your answers.)

   (a) $\int xe^{-x^2}dx$.

   Solution: Let $u = -x^2$. $du = -2xdx$.

$$\begin{aligned} & \int xe^{-x^2}dx \\ = & -\frac{1}{2}\int e^{-x^2}(-2x)\,dx \\ = & -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C \\ = & -\frac{1}{2}e^{-x^2} + C. \end{aligned}$$

□

   (b) $\int \frac{x}{1+x^4}dx$.

   Solution: Let $u = x^2$. $du = 2xdx$.

$$\begin{aligned} & \int \frac{x}{1+x^4}dx \\ = & \frac{1}{2}\int \frac{2x}{1+(x^2)^2}dx = \frac{1}{2}\int \frac{du}{1+u^2} \\ = & \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan\left(x^2\right) + C. \end{aligned}$$

□

   (c) $\int \frac{\ln x}{x^2}dx$.

   Solution: We use integration by parts. Let $u = \ln x$, $dv = \frac{dx}{x^2}$. $du = \frac{dx}{x}$. $v = -\frac{1}{x}$.

$$\begin{aligned} & \int \frac{\ln x}{x^2}dx \\ = & -\frac{1}{x}\ln x - \int\left(-\frac{1}{x}\right)\frac{dx}{x} \\ = & -\frac{1}{x}\ln x + \int \frac{dx}{x^2} \\ = & -\frac{1}{x}\ln x - \frac{1}{x} + C. \end{aligned}$$

□

2. (4 points)

   (a) Find the limit

$$\lim_{x\to 0} x^2 \ln x$$

   Solution: Since $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0}\ln x = -\infty$, this is an indeterminate form of type $0\cdot\infty$. We change it into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then apply l'Hospital's Rule:

$$\begin{aligned} \lim_{x\to 0} x^2\ln x &= \lim_{x\to 0}\frac{\ln x}{\frac{1}{x^2}} \quad \left(\frac{\infty}{\infty}\right) \\ &= \lim_{x\to 0}\frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x\to 0}\frac{1}{x}\cdot\frac{x^3}{-2} \\ &= \lim_{x\to 0}\frac{x^2}{-2} = 0. \end{aligned}$$

□

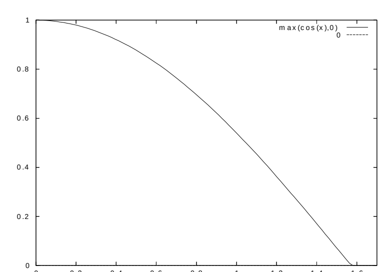
   (b) Find g' where

$$g(x) = \int_x^{x^2}\frac{\sin t}{t^2}dt$$

   Solution: We use the Fundamental Theorem of Calculus, Part 1:

$$\begin{aligned} g' &= \left(\int_x^{x^2}\frac{\sin t}{t^2}dt\right)' \\ &= \left(\int_x^1\frac{\sin t}{t^2}dt + \int_1^{x^2}\frac{\sin t}{t^2}dt\right)' \\ &= \left(-\int_1^x\frac{\sin t}{t^2}dt + \int_1^{x^2}\frac{\sin t}{t^2}dt\right)' \\ &= -\frac{\sin x}{x^2} + \frac{\sin\left(x^2\right)}{\left(x^2\right)^2}\cdot 2x. \end{aligned}$$

3. (4 points) Let $\mathcal{R}$ be the region in the plane bounded by the x axis and the curve $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ (see the figure). Find the volume of the solid obtained by rotating $\mathcal{R}$ around the y axis.



   Solution: We use the cylindrical shells method to calculate the volume:

$$\begin{aligned} V &= \int_0^{\frac{\pi}{2}} 2\pi x \cos x dx \\ &= 2\pi \left[x \sin x\right]_0^{\frac{\pi}{2}} - 2\pi \int_0^{\frac{\pi}{2}} \sin x dx \\ &= 2\pi \cdot \frac{\pi}{2} - 0 - 2\pi \left[-\cos x\right]_0^{\frac{\pi}{2}} \\ &= \pi^2 + 2\pi (0-1) = \pi^2 - 2\pi. \end{aligned}$$

Comment: If we have used the disk method, the integral would be

$$V = \int \pi \left(\arccos y\right)^2 dy$$

We would realize that this is not the best way and changed to the cylindrical shells method. □

4. (3 points) You operate a chocolate factory. The market price of pure dark chocolate is \$30 per kilogram, and at this price, you can sell as much chocolate as you want to produce. The cost of producing x kilograms of chocolate per day is $f(x) = x^3 + \frac{3}{2}x^2 + 12x + 1$. What quantity of chocolate should you produce per day, so as to maximize your profits?

   Recall: [profit] = [revenue]-[cost], and [revenue] = [price] x [quantity sold].

   Solution: Suppose that x kg of chocolate is produced per day. The revenue is $30x$. Thus the profit is

$$\begin{aligned} p(x) &= 30x - \left(x^3 + \frac{3}{2}x^2 + 12x + 1\right) \\ &= -x^3 - \frac{3}{2}x^2 + 18x - 1. \end{aligned}$$

$$\begin{aligned} p'(x) &= -3x^2 - 3x + 18 \\ &= -3\left(x^2 + x - 6\right) \\ &= -3(x+3)(x-2). \end{aligned}$$

   $p' = 0$ when $x = -3$ or $x = 2$. The domain of $p(x)$ is $[0, \infty)$. The critical numbers are 0 and 2. Since $p(2) > p(0)$, $x = 2$ the absolute maximum. Producing 2 kg chocolate per day will maximize the profits. □