MATH 1100 2008 Section A Midterm Test 1

NOVEMBER 4, 2008

Dr. Bing Zhou — Name ______________________________

1. (8 points) Find y'. (Do not simplify).

   (a) $y = \frac{e^x}{\sin x \cos x}$

   Solution:

$$y' = \frac{e^x \sin x \cos x - e^x (\cos x \cos x - \sin x \sin x)}{(\sin x \cos x)^2}.$$

$\square$

   (b) $y = \tan\left(\ln\left(1 + x^3\right)\right)$

   Solution:

$$y' = \sec^2\left(\ln\left(1 + x^3\right)\right) \cdot \frac{1}{1 + x^3} \cdot 3x^2.$$

$\square$

2. (4 points) Let $f(x) = \sqrt{1 + 3x}$. Use the 'limit' definition of the derivative to show that

$$f'(x) = \frac{3}{2\sqrt{1 + 3x}}.$$

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{1 + 3(x + h)} - \sqrt{1 + 3x}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{1 + 3(x + h)} - \sqrt{1 + 3x}}{h} \cdot \frac{\sqrt{1 + 3(x + h)} + \sqrt{1 + 3x}}{\sqrt{1 + 3(x + h)} + \sqrt{1 + 3x}} \\
&= \lim_{h \to 0} \frac{1 + 3(x + h) - (1 + 3x)}{h} \cdot \frac{1}{\sqrt{1 + 3(x + h)} + \sqrt{1 + 3x}} \\
&= \lim_{h \to 0} \frac{3h}{h} \frac{1}{\sqrt{1 + 3(x + h)} + \sqrt{1 + 3x}} = \frac{3}{2\sqrt{1 + 3x}}.
\end{aligned}$$

$\square$

3. (4 points) Find the horizontal and vertical asymptotes (if they exist) of the curve

$$y = \frac{9 \sin 5x}{x(x - 1)}.$$

Solution: Since

$$0 \le \left|\frac{9 \sin 5x}{x(x - 1)}\right| \le \left|\frac{9}{x(x - 1)}\right|$$

and

$$\lim_{x \to \infty} \left|\frac{9}{x(x - 1)}\right| = 0,$$

we have

$$\lim_{x \to \infty} \left|\frac{9 \sin 5x}{x(x - 1)}\right| = 0.$$

Therefore

$$\lim_{x \to \infty} \frac{9 \sin 5x}{x(x - 1)} = 0.$$

Similarly, we have

$$\lim_{x\to-\infty}\frac{9\sin 5x}{x\,(x-1)}=0.$$

Therefore, $y=0$ is the horizontal asymptote.

Since

$$\lim_{x\to 0}\frac{9\sin 5x}{x\,(x-1)}=\lim_{x\to 0}\frac{9\times 5}{(x-1)}\frac{\sin 5x}{5x}=-45,$$

$x=0$ is not a vertical asymptote. We note that $\pi<5<2\pi$, so $\sin 5<0$ and

$$\lim_{x\to 1^-}\frac{9\sin 5x}{x\,(x-1)}=\infty,$$

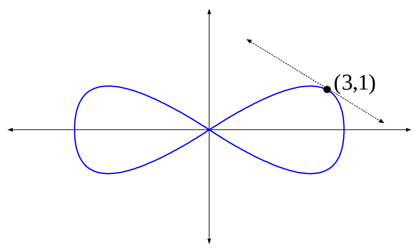
$$\lim_{x\to 1^+}\frac{9\sin 5x}{x\,(x-1)}=-\infty.$$

Therefore $x=1$ is the vertical asymptote. □

4. (4 points) The set of solutions to the equation

$$2(x^2+y^2)^2=25(x^2-y^2)$$

forms a curve called a *lemniscate*. Use implicit differentiation to find the slope of the lemniscate at the point $(x,y)=(3,1)$.



Solution: Suppose y is a function of x —that is $y=y(x)$. Differentiating both sides of the lemniscate equation, we get

$$4(x^2+y^2)\cdot\Big(2x+2y\cdot y'\Big) \quad = \quad 25\Big(2x-2y\cdot y'\Big),$$

which simplifies to

$$8x(x^2+y^2)+8(x^2+y^2)\cdot y\cdot y' \quad = \quad 50x-50y\cdot y',$$

which simplifies to

$$\Big(50+8(x^2+y^2)\Big)\cdot y\cdot y' \quad = \quad 50x-8x(x^2+y^2)$$

which simplifies to

$$y' \quad = \quad \frac{50x-8x(x^2+y^2)}{\Big(50+8(x^2+y^2)\Big)\,y}.$$

Thus, if $(x,y)=(3,1)$, then

$$\begin{aligned}\text{slope} &= \frac{50x-8x(x^2+y^2)}{\Big(50+8(x^2+y^2)\Big)\,y} = \frac{50(3)-8(3)((3)^2+(1)^2)}{\Big(50+8((3)^2+(1)^2)\Big)\,(1)}\\ &= \frac{150-24(9+1)}{50+8(9+1)} = \frac{-90}{130} = \boxed{\tfrac{-9}{13}.}\end{aligned}$$

□