Mathematics 110 – Calculus of one variable
TRENT UNIVERSITY, 2001-2002

Test #2
Friday, 8 February, 2002
Time: 50 minutes

1. Compute any *three* of the integrals **a-e**. *[12 = 3 × 4 ea.]*

$$\textbf{a.}\ \int_{-\pi/2}^{\pi/2} \cos^3(x)\,dx \qquad \textbf{b.}\ \int x^2 \ln(x)\,dx \qquad \textbf{c.}\ \int_0^1 \left(e^x\right)^2\,dx$$

$$\textbf{d.}\ \int \frac{e^{2x}\ln\left(e^{2x}+1\right)}{e^{2x}+1}\,dx \qquad \textbf{e.}\ \int_1^e \left(\ln(x)\right)^2\,dx$$

Solutions.

a.

$$\int_{-\pi/2}^{\pi/2} \cos^3(x)\,dx = \int_{-\pi/2}^{\pi/2} \cos^2(x)\cos(x)\,dx = \int_{-\pi/2}^{\pi/2} \left(1-\sin^2(x)\right)\cos(x)\,dx$$

We'll substitute $u = \sin(x)$, so $du = \cos(x)\,dx$, $-1 = \sin(-\pi/2)$, and $1 = \sin(\pi/2)$.

$$= \int_{-1}^{1} \left(1-u^2\right)\,du = \left(u - \frac{u^3}{3}\right)\Big|_{-1}^{1} = \left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right) = \frac{4}{3} \quad \blacksquare$$

b. We'll use integration by parts, with $u = \ln(x)$ and $dv = x^2\,dx$, so $du = \frac{1}{x}\,dx$ and $v = \frac{x^3}{3}$.

$$\int x^2\ln(x)\,dx = \frac{x^3}{3}\ln(x) - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx = \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C \quad \blacksquare$$

c. After bit of algebra, we'll use the substitution $u = 2x$, so $du = 2\,dx$ (and $\frac{1}{2}\,du = dx$), $0 = 2\cdot 0$, and $2 = 2\cdot 1$.

$$\int_0^1 \left(e^x\right)^2\,dx = \int_0^1 e^{2x}\,dx = \int_0^2 e^u\cdot\frac{1}{2}\,du = \frac{1}{2}e^u\Big|_0^2 = \frac{1}{2}\left(e^2-1\right) \quad \blacksquare$$

d. We'll substitute whole hog: let $w = \ln\left(e^{2x}+1\right)$, so $dw = \frac{2e^{2x}}{e^{2x}+1}\,dx$ (and $\frac{1}{2}\,dw = \frac{e^{2x}}{e^{2x}+1}$).

$$\int \frac{e^{2x}\ln\left(e^{2x}+1\right)}{e^{2x}+1}\,dx = \int w\cdot\frac{1}{2}\,dw = \frac{w^2}{4} + C = \frac{1}{4}\left(\ln\left(e^{2x}+1\right)\right)^2 + C \quad \blacksquare$$

e. We'll use integration by parts, with $u = \left(\ln(x)\right)^2$ and $dv = dx$, so $du = 2\ln(x)\cdot\frac{1}{x}\,dx$ and $v = x$.

$$\int_1^e \left(\ln(x)\right)^2\,dx = x\left(\ln(x)\right)^2\Big|_1^e - \int_1^e x\cdot 2\ln(x)\cdot\frac{1}{x}\,dx = \left(e\cdot 1^2 - 1\cdot 0^2\right) - 2\int_1^e \ln(x)\,dx$$

We use integration by parts again, with $u = \ln(x)$ and $dv = dx$, so $du = \dfrac{1}{x}\,dx$ and $v = x$.

$$= e - 2\left(x\ln(x)\big|_1^e - \int_1^e x\cdot\frac{1}{x}\,dx\right) = e - 2\left((e\cdot 1 - 1\cdot 0) - \int_1^e 1\,dx\right)$$

$$= e - 2\left(e - x\big|_1^e\right) = e - 2\left(e - (e-1)\right) = e - 2 \quad \blacksquare$$

2. Do any *two* of **a**-**c**. *[8 = 2 × 4 ea.]*

a. Compute $\int_0^1 (2x+3)\,dx$ using the Right-hand Rule.

b. Compute $\frac{dy}{dx}$ if $y = \int_0^{x^2} \sqrt{t}\,dt$ (where $x \geq 0$) without evaluating the integral.

c. Compute $\int_{-1}^{1} \sqrt{1-x^2}\,dx$ by interpreting it as an area.

Solutions.

a. If we partition $[0,1]$ into n equal subintervals, then the ith subinterval is $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, which has width $\frac{1}{n}$ and right endpoint $\frac{i}{n}$. Thus the area of the ith rectangle in the Right-hand Rule Riemann sum is $\left(2\frac{i}{n}+3\right)\frac{1}{n}$. Hence

$$\begin{aligned}
\int_0^1 (2x+3)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2\frac{i}{n}+3\right)\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(2\frac{i}{n}+3\right) \\
&= \lim_{n\to\infty} \frac{1}{n}\left[2\left(\sum_{i=1}^{n}\frac{i}{n}\right) + \left(\sum_{i=1}^{n} 3\right)\right] = \lim_{n\to\infty} \frac{1}{n}\left[\frac{2}{n}\left(\sum_{i=1}^{n} i\right) + 3n\right] \\
&= \lim_{n\to\infty} \frac{1}{n}\left[\frac{2}{n}\cdot\frac{n(n+1)}{2} + 3n\right] = \lim_{n\to\infty} \frac{1}{n}\left[(n+1)+3n\right] = \lim_{n\to\infty} \frac{1}{n}\left[4n+1\right] \\
&= \lim_{n\to\infty} \left[\frac{4n}{n} + \frac{1}{n}\right] = \lim_{n\to\infty} \left[4 + \frac{1}{n}\right] = 4 + 0 = 4 \quad \blacksquare
\end{aligned}$$

b. Let $u = x^2$; since $x \geq 0$, $x = \sqrt{u}$. Then, using the Chain Rule and the Fundamental Theorem of Calculus,

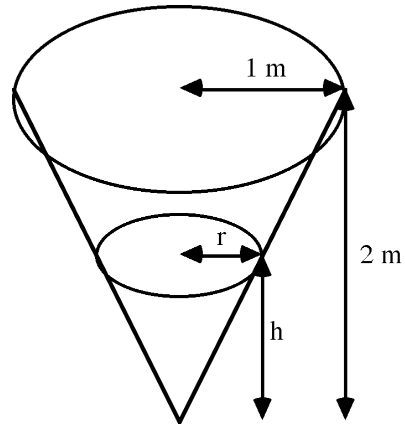
$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left(\frac{d}{du}\int_0^u \sqrt{t}\,dt\right)\cdot\frac{du}{dx} = \sqrt{u}\cdot\frac{du}{dx} = \sqrt{x^2}\cdot\frac{d}{dx}x^2 = x\cdot 2x = 2x^2 \quad \blacksquare$$

c. Note that $y = \sqrt{1-x^2}$, $-1 \leq x \leq 1$, is the upper half of the unit circle $x^2+y^2=1$. This circle has area $\pi 1^2 = \pi$, so $\int_{-1}^{1} \sqrt{1-x^2}\,dx$, which represents the area of the upper half of the circle, is equal to $\frac{\pi}{2}$. $\blacksquare$

3. Water is poured at a rate of 1 m^3/min into a conical tank (set up point down) 2 m high and with radius 1 m at the top. How quickly is the water rising in the tank at the instant that it is 1 m deep over the tip of the cone? *[8]*

(The volume of a cone of height h and radius r is $\frac{1}{3}\pi r^2 h$.)

Solution. At any given instant, the water in the tank occupies a conical volume, with height – that is, depth in the tank – h and radius r in the same proportions as the tank as a whole.



Hence $\frac{r}{h} = \frac{1}{2}$, so $r = \frac{h}{2}$, and it follows that the volume of the water at the given instant is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3 .$$

Note that the rate at which the water is rising in the tank is $\frac{dh}{dt}$.

Since, on the one hand $\frac{dV}{dt} = 1\ m^3/min$, and on the other hand

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{12}\pi h^3\right) = \frac{d}{dh}\left(\frac{1}{12}\pi h^3\right) \cdot \frac{dh}{dt} = \frac{1}{12}\pi \cdot 3h^2 \cdot \frac{dh}{dt} = \frac{1}{4}\pi h^2 \cdot \frac{dh}{dt} ,$$

we know that any given instant, $\frac{dh}{dt} = \frac{dV}{dt} / \frac{1}{4}\pi h^2 = 1/\frac{1}{4}\pi h^2 = 4/\pi h^2$. At the particular instant that $h = 1\ m$, it follows that $\frac{dh}{dt} = 4/\pi 1^2 = 4/\pi\ m/min$. ■

4. Consider the region in the first quadrant with upper boundary $y = x^2$ and lower boundary $y = x^3$, and also the solid obtained by rotating this region about the y-axis.

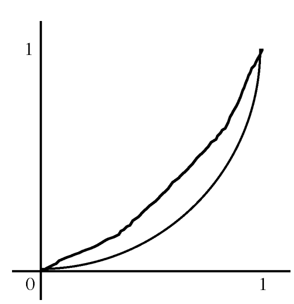
a. Sketch the region and find its area. *[4]*

b. Sketch the solid and find its volume. *[7]*

c. What is the average area of either a washer or a shell (your pick!) for the solid? *[1]*

Solution.

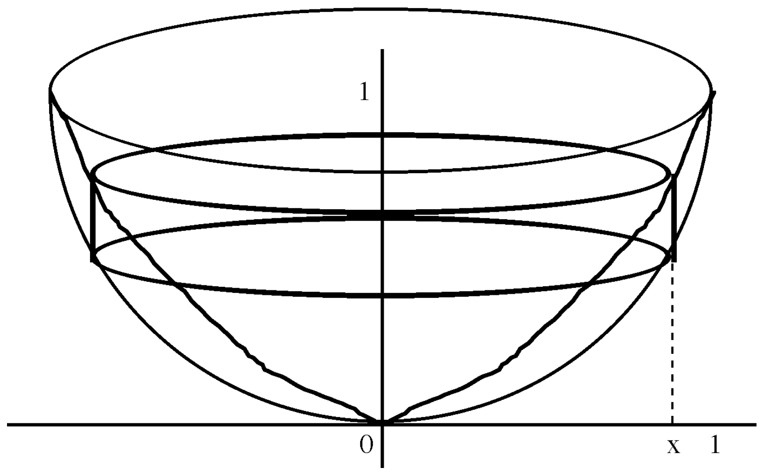
a. First, we find the points of intersection of the two curves: if $x^2 = x^3$, then $x = 0$ or $x = x^3/x^2 = 1$. Note that when $0 \le x \le 1$, then $x^3 = x^2 \cdot x \le x^2 \cdot 1 = x^2$. It's not too hard to see that the region between the curves looks more or less like:



The area of the region is then

$$\int_0^1 \left(x^2 - x^3\right)\, dx = \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \left(\frac{1}{3} - \frac{1}{4}\right) - (0 - 0) = \frac{1}{12} \quad \blacksquare$$

b. Rotating (revolving, whatever ...) the region about the y-axis produces the following solid.



The volume of this solid is a little easier to compute using shells than using washers. Since we rotated the region about a vertical line, we will use x as the variable of integration; note that $0 \leq x \leq 1$ over the region in question. With respect to x, a generic cylindrical shell has radius $r = x - 0 = x$ and height $h = x^2 - x^3$. Thus the volume of the solid is

$$\begin{aligned}\int_0^1 2\pi rh\, dx &= \int_0^1 2\pi x\left(x^2 - x^3\right)\, dx = 2\pi \int_0^1 \left(x^3 - x^4\right)\, dx \\ &= 2\pi\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1 = 2\pi\left[\left(\frac{1}{4} - \frac{1}{5}\right) - (0 - 0)\right] = 2\pi\frac{1}{20} = \frac{\pi}{10}\,. \quad \blacksquare\end{aligned}$$

c. From **b** we know that the area of the cylindrical shell for x, where $0 \leq x \leq 1$, is $2\pi x\left(x^2 - x^3\right)$. Thus the average area of a cylindrical shell for this solid is

$$\frac{1}{1-0}\int_0^1 2\pi x\left(x^2 - x^3\right)\, dx = 1 \cdot \frac{\pi}{10} = \frac{\pi}{10}\,. \quad \blacksquare$$