Mathematics 110 – Calculus of one variable
Trent University 2001-2002

ASSIGNMENT #10
Due: Friday, 29 March, 2002

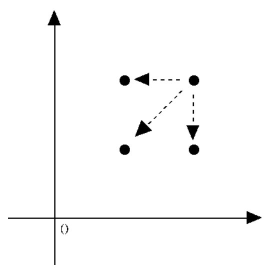
Splitter, Mover, Triangle-Maker

Define a function F as follows:

- The domain and range of F is the collection of all the non-empty subsets of the xy-plane.
- If A is a subset of the xy-plane, then

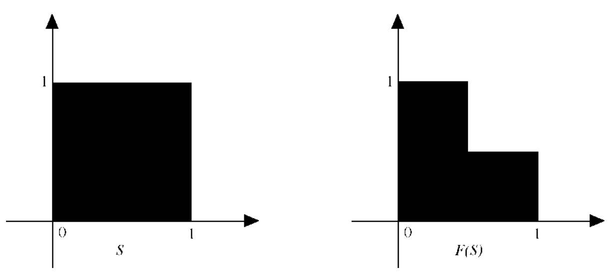
$$F(A) = \left\{ \left(\frac{x}{2}, \frac{y}{2}\right) \mid (x, y) \in A \right\} \cup \left\{ \left(\frac{x+1}{2}, \frac{y}{2}\right) \mid (x, y) \in A \right\} \cup \left\{ \left(\frac{x}{2}, \frac{y+1}{2}\right) \mid (x, y) \in A \right\}$$

Roughly speaking, what F does is take each point in A and splits it into three points, one halfway to the x-axis, one halfway to the y-axis, and one halfway to the origin from the original point:



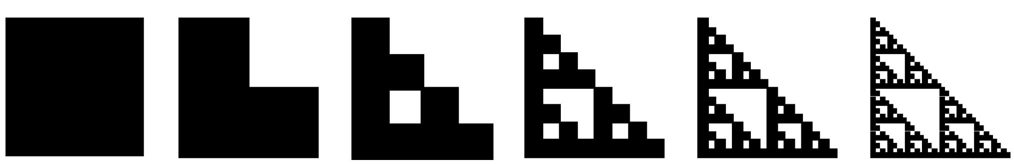
[Actually, this description isn't quite correct — it is true of the point $(1, 1)$*, but no other — but it is true that* F *splits each point in* A *into three points, one of which is halfway to the origin from the original point.]*

For example, consider what F does with the unit square, $S = \{ (x, y) \mid 0 \le x \le 1 \ \&\ 0 \le y \le 1 \}$:



1. Sketch $F(F(S))$, $F(F(F(S)))$, $F(F(F(F(S))))$, and $F(F(F(F(F(S)))))$. (Separately!) *[2]*

Solution. Here are S, $F(S)$, ... , $F(F(F(F(F(S)))))$:



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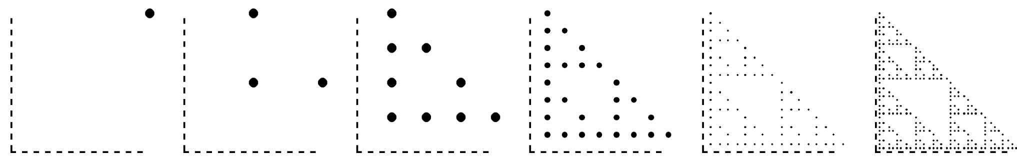
2. Let $P = \{ (1,1) \}$. Sketch P, $F(P)$, $F(F(P))$, $F(F(F(P)))$, $F(F(F(F(P))))$, and $F(F(F(F(F(P)))))$. (Separately!) *[2]*

Solution. Here are P, $F(P)$, ..., $F(F(F(F(F(P)))))$:



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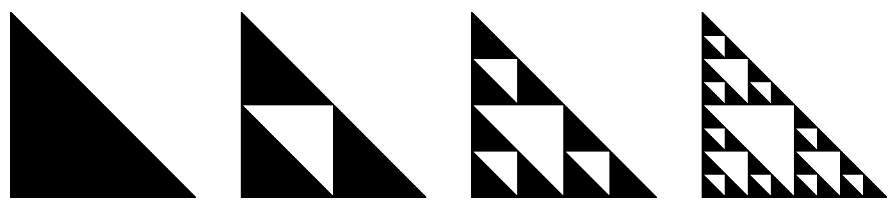
3. What shape do you get as the limit of the sequence A, $F(A)$, $F(F(A))$, $F(F(F(A)))$, ..., no matter what non-empty subset A of the first quadrant you start with? Give a rough sketch. *[3]*

Solution. The last picture in the solution of either **1** or **2** will serve for a rough sketch. The shape is one obtained by a process much like that for making the Cantor set: Start with a solid triangle, remove the middle triangle from it, leaving three smaller triangles; remove the middle triangle from each of the remaining triangles; repeat the last until the end of time ... Here are the first four shapes along the way:



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4. Where on Trent's web pages is there (a very slightly modified version of) this shape? *[1]*

Solution. Check out the logo of the Department of Mathematics, *e.g.* at:

`http://www.trentu.ca/mathematics/`

It's in the upper left corner of the page ... ■

5. What kind of object is this shape? What is this particular one called? *[2]*

Solution. The shape is a *fractal* called *Sierpinski's Triangle*. Roughly speaking, fractals are shapes in which parts of the shape are scaled versions of the whole shape. The snowflake curve of Problem 2 on Assignment #3 is another example of a fractal. ■