On color critical graphs with large adapted chromatic numbers

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Abstract

When the vertices and edges are colored with k colors, an edge is called monochromatic if the edge and the two vertices incident with it all have the same color. The adapted chromatic number of a graph G, $\chi_a(G)$, is the least integer k such that for each k-edge coloring of G the vertices of G can be colored with the same set of colors without creating any monochromatic edges. It is easy to see that $\chi_a(G) \leq \chi(G)$. While this bound is tight, all the known graphs attainding this bound are not color critical. It is known that if G is a critical graph, then $\chi_a(G) \leq \chi(G) - 1$. In this article we construct a family of k-critical graphs whose adapted chromatic number is exactly one less than their chromatic number. This answers a question in [Molloy, M., & Thron, G. (2012). An asymptotically tight bound on the adaptable chromatic number. Journal of Graph Theory, 71(3), 331–351. doi:10.1002/jgt.20649]. We also study the properties of graphs that are critical with respect to adaptable chromatic number.

1 Introduction

For notation and graph theory terminology, we in general follow [14]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. A graph $H = (W, F)$ is a subgraph of G if $W \subseteq V$ and $F \subseteq E$. If $F = \{xy \in E : x \in W, y \in W\}$, then H is a subgraph of G induced by W, and we write $H = G[W]$. Similarly, if $W = \{x : xy \in F$ for some vertex $y.\}$, then H is a subgraph induced by F and we write $H = G[F]$. Let $x \in V(G)$ and $e \in E(G)$. We also write $G[V(G) \backslash x]$ as $G - x$ and $G[E(G) \backslash e]$ as $G - e$.

When the vertices and edges of G are colored with colors from a set S, an edge is called monochromatic if the edge and the two vertices incident with it all have the same color. A vertex coloring C is called adapted to an edge coloring C' if C and C' together do not produce any monochromatic edge. A graph G is adaptably k-colorable if when $|S| = k$, for every edge coloring C' using colors in S, there is a vertex coloring C using colors in S that is adapted to C'. The adaptable chromatic number of G, denoted $\chi_a(G)$, is the least k such that

G is adaptably k-colorable. In the literature, the largest k such that G is not adaptably k-colorable is called the chromatic capacity of G, denoted $\chi_{CAP}(G)$. Thus

$$\chi_{CAP}(G) = \chi_a(G) - 1.$$

The term color capacity was first used by Archer in [1]. Independently, Hell and Zhu defined the adaptable chromatic number in [8]. Many results have been proved independently by different authors using either term (see e.g. [4], [5], [7], [9], [10], [11], [12], [13], [15], [16]).

To understand the adaptable chromatic number of a graph, we find it more intuitive to consider a coloring game played on a graph G. Player E (the edge colorer) will color the edges of G first using colors in S. After all the edges are colored, player V (the vertex colorer) will color the vertices with the same set of colors. If player V can color all the vertices without creating any monochromatic edges, he wins the game; otherwise player E wins. The largest number of colors such that player E has a winning strategy is the chromatic capacity of G, and the least number of colors such that player V always has a winning strategy (that strategy may depend on the edge coloring of player E's) is the adaptable chromatic number of G.

Recall that the adaptable chromatic number is the least number of colors one needs to color the vertices while not creating any monochromatic edges after the edges are coloured. It is obvious that we have

$$\chi_a(G) \le \chi(G) \tag{1}$$

since a proper vertex coloring will not create any monochromatic edges regardless how the edges are colored. For graphs in general, this upper bound for $\chi_a(G)$ is sharp. There are a number of ways to construct graphs with their adapted chromatic number the same as their chromatic number ([5] and [8]). However, as noted in [13], for a color critical graph G, that is, a graph G such that $\chi(G-e) < \chi(G)$ for every edge e in G, we have $\chi_a(G) \le \chi(G) - 1$. The authors asked whether this bound is sharp: Are there any color critical graph G such that $\chi_a(G) \le \chi(G) - 1$? In Section 2, we describe a method to construct such critical graphs. In Section 3 we define adaptably critical graphs and discuss the properties of these graphs. In Section 4, we pose some problems involving adaptable chromatic number and critical graphs.

2 k-critical graph G with $\chi_a(G) = \chi(G) - 1$

A graph G is color critical if $\chi(H) < \chi(G)$ whenever H is a proper subgraph of G. Equivalently, G is color critical (or critical) if $\chi(G-e) = \chi(G) - 1$ for every edge e in G. It is proved in [8] that $\chi_a(G) \le \chi(G - E')$ if E' is a set of edges in G and $\chi(G - E') > |E'|$. When we choose $E' = \{e\}$ in a critical graph G, this implies $\chi_a(G) \le \chi(G) - 1$. Molloy and Thron asked ([13], Question 4 and 5) whether there are critical graphs that attain this bound. We will give a positive answer to their questions and show this bound is sharp by presenting a construction for critical graphs G such that $\chi_a(G) = \chi(G) - 1$.

The method we use to construct k-critical graphs with high adaptable chromatic number will use the join operation of two graphs and the Hajós' construction. The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is defined by

$$V(G_1 \vee G_2) = V(G_1) \cup V(G_2),$$
$$E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}.$$

In other words, we construct $G_1 \vee G_2$ by taking a copy of each of G_1 and G_2 and joining every vertex in G_1 with every vertex in G_2. For example, let $G_1 = K_1$ and $G_2 = C_5$ the $G_1 \vee G_2 = W_5$. (See Figure 1.) If G_2 is a single vertex x, we also write $G_1 \vee G_2$ as $G_1 \vee x$. The Hajós' construction [6] for two disjoint graphs G_1, G_2 and edges $x_1y_1 \in E(G_1)$, $x_2y_2 \in E(G_2)$ is obtained by removing x_1y_1 and x_2y_2, identifying x_1 and x_2, and joining y_1 and y_2 by a new edge. For example, for the graphs G_1 and G_2 in figures 1 and 2, the result when Hajós' construction is applied to them is the graph is Figure 3.

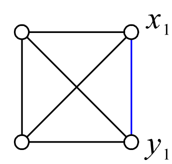

Figure 1: G_1

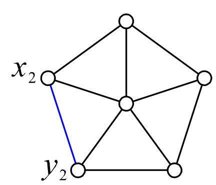

Figure 2: G_2

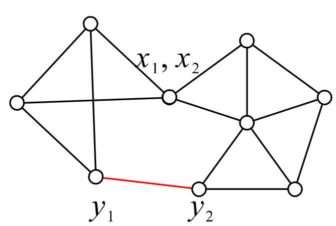

Figure 3: Hajós' construction applied to G_1 and G_2.

The following results are well known: If G_1 is a k_1-critical graph and G_2 a k_2-critical graph, then the join $G_1 \vee G_2$ is a $(k_1 + k_2)$-critical graph. If both G_1 and G_2 are k-critical graphs, then the graph obtained by applying the construction of Hajós is also k-critical. We will construct a k-color critical graph G_k for every integer $k \geq 3$ that contains a subgraph H_k such that $\chi_a(H_k) \geq k-1$. This implies $\chi_a(G_k) = k-1$.

Theorem 1 *For every integer k such that $k \geq 3$, there is a k-critical graph G_k such that it contains a subgraph H_k, $\chi_a(H_k) \geq k-1$, and $V(H_k)$ is a proper subset of $V(G_k)$.*

Proof. We can choose any 3-critical graph, an odd cycle, as G_3 and H_3 is K_2.

We construct G_k recursively for $k \geq 4$.

Construction 1 Let u be a vertex in $V(G_{k-1}) \setminus V(H_{k-1})$. Since G_{k-1} is $(k-1)$-critical and $k \geq 4$, $\deg(u) \geq k-2 \geq 2$. Let x and y be two neighbors of u. Notice that neither (u,x) or (u,y) is an edge in H_{k-1}. We take $2(k-2)-1$ copies of G_{k-1} and label the special vertices $u^{(i)}$, $x^{(i)}$, and $y^{(i)}$, $i = 1, 2, ..., 2(k-2)-1$. We identify the vertices $u^{(1)}, u^{(2)}, ..., u^{(k-2)}$, remove the edges $\left(u^{(i)}, x^{(i)}\right)$ for $i = 2, ..., k-2$ and $\left(u^{(j)}, y^{(j)}\right)$ for $j = 1, 2, ..., 2(k-2)-2$, and add the edges $\left(y^{(i)}, x^{(i+1)}\right)$ for $i = 1, 2, ..., 2(k-2)-2$. We denote the resulted graph F_k. F_k contains $k-2$ disjoint copies of H_{k-1}. Finally, we add a new vertex w and let $G_k = F_k \vee w$.

F_k can be viewed as the result of a sequence of Hajós operations applied to $(k-1)$-critical graphs isomorphic to G_{k-1}. Therefore, F_k is $(k-1)$-critical and $G_k = K_1 \vee F_k$ is k-critical.

Let H_k be the subgraph of G_k that is the join of w with the $k-2$ disjoint copies of H_{k-1}. Since $V(H_{k-1})$ is a proper subset of $V(G_{k-1})$, $V(H_k)$ is a proper subset of $V(H_k)$. We will show that $\chi_a(H_k) \geq k-1$ by showing Player E has a winning strategy if $k-2$ colors are used.

Let the color set be $\{1, 2, ..., k-2\}$. Since $\chi_a(H_{k-1}) \geq k-2$, Player E has a winning strategy on H_{k-1} when $k-3$ colors are used. Player E colors the edges between w and the vertices of the ith disjoint copy of H_{k-1} with color i and the edges in the ith copy of H_{k-1} according to the winning strategy using colors $\{1, ..., k-2\} \setminus \{i\}$. We claim this is a winning strategy for Player E. Suppose that Player V colors the vertices of H_k with colors $\{1, 2, ..., k-2\}$ and w is assigned color j. Since all the edges between w and the jth copy of H_k are colored with color j, no vertex in that copy of H_k can be colored with color j. Player V has to color that copy of H_k with colors $\{1, 2, ..., k-2\} \setminus \{j\}$. Since Player E has colored the edges using a winning strategy for these colors, there will be a monochromatic edge.

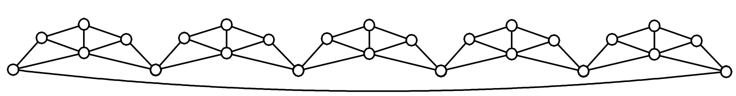

Figure 4: F_5

■

For example, if we use C_3 as G_3, then $F_4 = C_5$ and $G_4 = W_5$, the wheel with five vertices in the rim. The graphs F_5 and G_5 are in figures 4 and 5. A

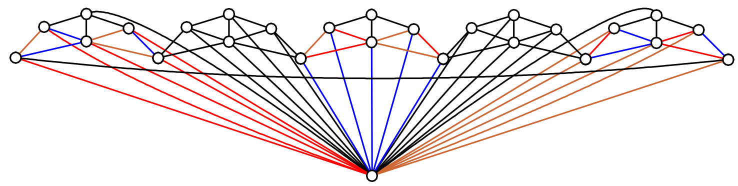

Figure 5: G_5

winning strategy for player E using three colors (red, blue and brown, the black edges can be colored arbitrarily) is also given in Figure 5.

3 Adaptably critical graphs

The graphs constructed in the proof of Theorem 1 contains a proper subgraph that has the same adaptable chromatic number as the whole graph. These graphs are not critical with respect to adaptable coloring. In this section, we investigate the graphs that are critical with respect to adaptable coloring.

Definition 2 *A graph G is adaptably critical if for every proper subgraph H of G, $\chi_a(H) < \chi_a(G)$.*

Equivalently, G is adaptably critical if for every edge e in G, $\chi_a(G-e) < \chi_a(G)$. If $\chi_a(G) = k$ and G is adaptably critical, we say G is k-adaptably critical. It is obvious that K_2 is the only 2-adaptably critical graph. The graphs that are adaptably 2-colorable were characterized in [8]. An odd edge-bicycle is the union of two edge-disjoint odd cycles and a path (also edge-disjoint from the cycles but may have length zero) joining the two cycles. An odd edge-K_4 is a graph obtained by subdividing the edges of a K_4 such that the length of each of the four cycles is odd.

Theorem 3 *$\chi_\alpha(G) \leq 2$ if and only if G does not have an odd edge-bicycle or an odd edge-K_4.*

Using this result, we can characterize the 3-adaptably critical graphs.

Corollary 4 *A graph G is 3-adaptably critical if and only if G is either an odd edge-bicycle or an odd edge-K_4.*

As a consequence of Corollary 4, K_4 is the only 3-adaptably critical graph that is also color critical. It would be interesting to know if there are any other graphs that are both critical and adaptably critical.

In the early stages of the studies of color critical graphs, Dirac ([2], [3]) proved that for every k-critical graph G, $\delta(G) \geq k-1$ and this bound is sharp. This inequality also holds for adaptably critical graphs.

Theorem 5 *If G is adaptably critical, then*

$$\delta(G) \geq \chi_a(G) - 1. \tag{2}$$

Proof. Suppose that G is a counter-example with $\chi_a(G) = k$ and $\delta(G) \leq k-2$. We derive a contradiction by showing that G is adaptably $(k-1)$-colorable. Let u be a vertex such that $\deg(u) = \delta(G)$. Since G is adaptably critical, $G - u$ is adaptably $(k-1)$-colorable. A winning strategy of player V on the graph $G - u$ with $k-1$ colors can be extended to the graph G by assigning a color that has not been used on any neighbors of u to u. ■

To see that the inequality (2) is also sharp, we will construct adaptably critical graphs G with $\delta(G) \geq \chi_a(G) - 1$. Unlike in the case of critical graphs, when the join operations or the Hajós constructions are applied to adaptably critical graphs, the result is not adaptably critical. However, the following construction used in [5] and [8] showing that there are arbitrarily large graphs G such that $\chi_\alpha(G) = \chi(G)$ will produce adaptably critical graphs when smaller adaptably critical graphs are used in the construction.
Construction 2 For $k \geq 2$, let G be a graph such that $\chi_\alpha(G) = k - 1$. Let G_i $(i = 1, 2, ..., k-1)$ be $k - 1$ disjoint copies of G. Let x be a new vertex. $H = \{x\} \vee \left(\cup_{i=1}^{k-1} G_i\right)$.

Theorem 6 *The result of Construction 2 is a k-adaptably critical graph H if the graph G is $(k-1)$-adaptably critical.*

Proof. It is known that $\chi_\alpha(H) = k$ ([5], [8]). We will show that $\chi_\alpha(H - e) = k-1$ for every edge $e \in E(H)$. Suppose that the color set is $S = \{1, 2, ..., k-1\}$ and player E has colored the edges of H using colors in S.

First we observe that player V has a winning strategy (even without the removal of e) if there is a color j^* such that for all $i \in \{1, 2, ..., k-1\}$ there is an edge between x and G_i that is not colored by color j^*. Suppose such j^* exists and xv_1, xv_2, ..., xv_{k-1} $(v_i \in G_i)$ are edges that are colored with colors other than j^*. Player V will color the vertices x and v_i $(i = 1, 2, ..., k-1)$ with color j^* and color $G_i - v_i$ using a winning strategy with color set $S \setminus \{j^*\}$. Since G_i is $(k-1)$-critical, such a strategy exists and there is no monochromatic edge in $G_i - v_i$. Since v_i is the only vertex in G_i with color j^*, there is no monochromatic edge in G_i. The only vertices with the same color, j^*, as x are $v_1, v_2, ..., v_{k-1}$ and the edge between x and any one of them is not colored with j^*. Therefore there is no monochromatic edge in this coloring of H.

We may assume then for each color $j \in S$ there is a $G_{j'}$ such that all the edges between x and $G_{j'}$ are colored with color j. Without loss of generality, we assume that all the edges between x and G_i are colored with color i $(i = 1, 2, ..., k-1)$. By symmetry, we only need to consider two special cases:
Case 1: $e = xv_1$ where $v_1 \in G_1$. Player V will color x and v_1 with color 1, color the vertices in $G_1 - v_1$ according to the winning strategy with color set $S \setminus \{1\}$ and color the vertices in G_i $(i \geq 2)$ according to the winning strategy with color set S. Since G_i is $(k-1)$-critical for each i, these strategies exist and there will be no monochromatic edge.

Case 2: $e = u_1v_1$ where $u_1, v_1 \in G_1$. Player V will color x with color 1, color the vertices in $G_1 - e$ according to the winning strategy with color set $S \setminus \{1\}$ and color the vertices in G_i $(i \geq 2)$ according to the winning strategy with color set S. Since G_i is $(k-1)$-critical for each i, these strategies exist and there will be no monochromatic edge. ■

If we let H_2 be K_2 and H_k be the result when Construction 2 is applied using H_{k-1}, then $\chi_a(H_k) = k$. H_k is adaptably critical and $\delta(H_k) = k-1$. Therefore the bound in (2) is sharp.

4 Concluding remarks

There are many questions about adaptable chromatic number which remain unanswered. The most interesting problem involving adaptable chromatic number is its relation to the chromatic number. The adaptable chromatic number of graph G is bounded above by the chromatic number of G according to (1) and this bound is sharp. In [7] it is shown that there are graphs G with arbitrarily large girth such that $\chi_a(G) = \chi(G)$. A method to construct such graphs is given in [15]. On the other hand, it is proved in [16] that there is a positive constant K such that for every graph G,

$$\chi_a(G) \geq K \log\log(\chi(G)). \tag{3}$$

For all known examples, the adaptable chromatic number of a graph G, $\chi_a(G)$, is at least the order of $\sqrt{\chi(G)}$. The following question is asked in [13].

Problem 7 *Are there any graphs G such that $\chi_a(G)$ is less than the order of $\sqrt{\chi(G)}$?*

Also we would like to ask

Problem 8 *Can the lower bound of $\chi_a(G)$ in (3) be improved?*

The graphs we constructed in the proof of Theorem 1 are not adaptably critical. A natural question is whether there are graphs that are critical with respect to both adaptable coloring and ordinary coloring.

Problem 9 *Are there any critical graphs except K_4 that are adaptably critical?*

The graphs we constructed in the proof of Theorem 1 also have large clique numbers. There are triangle-free k-critical graphs with adaptable chromatic number $k-1$. The Grötzsch graph is such an example. It is well-known that the Grötzsch graph is 4-critical. It is also easy to verify that Grötzsch graph has adaptable chromatic number three.

Problem 10 *For every positive integer $k \geq 5$, does there exist a triangle-free k-critical graph G such that $\chi_a(G) = k-1$?*

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