On a construction of graphs with high chromatic capacity and large girth*

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Abstract

The chromatic capacity of a graph G, $\chi_{CAP}(G)$, is the largest integer k such that there is a k-colouring of the edges of G such that when the vertices of G are coloured with the same set of colours, there are always two adjacent vertices that are coloured with the same colour as that of the edge connecting them. It is easy to see that $\chi_{CAP}(G) \leq \chi(G) - 1$. In this note we present a construction based on the idea of classic construction due to B. Descartes for graphs G such that $\chi_{CAP}(G) = \chi(G) - 1$ and G does not contain any cycles of length less than q for any given integer q.

1 Introduction

The term chromatic capacity was first introduced by Archer in [1]. The concept was used independently by several other authors around that time. Please see [5] and [6] for a summary of results on this subject. The definition of the chromatic capacity of a graph G, denoted $\chi_{CAP}(G)$, can be phrased in terms of a graph game. There are two players, A and B, in this game. Player A will colour the edges of G using colours $\{1, 2, ..., k\}$ first. After A

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finishes, B will colour the vertices of G using the same set of colours. If two adjacent vertices are coloured with the same colour as that of the edge connecting them, that edge is called a monochromatic edge. B wins the game if he can colour the vertices of G without creating any monochromatic edges. Otherwise A wins. The chromatic capacity of G is the largest integer k such that the player A has a winning strategy when the game is played with k colours. Or equivalently, the chromatic capacity of G is the largest integer k such that there is a k-colouring of the edges of G such that when the vertices of G are coloured with the same colours, there is always at least one monochromatic edge.

To see that the chromatic capacity of a graph is well-defined, we note that if $k \geq \chi(G)$, colouring the vertices of G properly would be a winning strategy of B, no matter how the edges were coloured. Therefore, the number of colours player A can have for him to have a winning strategy is bounded by $\chi(G) - 1$. That is,

$$\chi_{CAP}(G) \leq \chi(G) - 1. \tag{1}$$

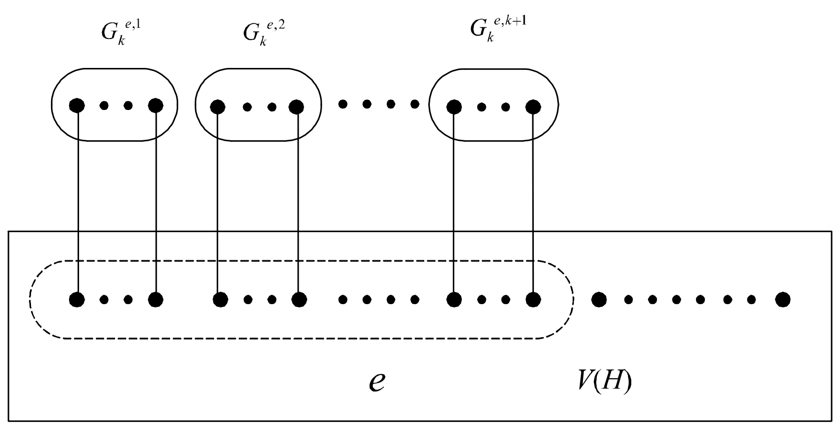
For some graphs, the bound in (1) can be reached. There are methods of constructing graphs G such that $\chi_{CAP}(G) = \chi(G) - 1$. In [5], Greene described methods to construct r-uniform hypergraphs and triangle-free graphs with this property. In [6], Huizenga constructed graphs G with the property $\chi_{CAP}(G) = \chi(G) - 1$ and no odd cycle of length less than q for any positive integer q. In both [5] and [6], it was asked if it is possible to construct graphs G such that $\chi_{CAP}(G) = \chi(G) - 1 = k$ and G does not contain any cycle of length less than q for any given positive integers k and q. In this paper, we present such a construction.

2 The construction of graphs with $\chi_{CAP} = \chi - 1$ and girth $\geq q$

Our construction is based on the idea of the famous construction due to B. Descartes [2][3] with a modification similar to the method described by Kostochka and Nešetřil in [7]. A hyper graph H consists of a vertex set $V(H)$ and an edge set $E(H)$. The elements of $E(H)$, the edges of H, are nonempty subsets of $V(H)$. H is r-uniform if all its edges have size r. The chromatic number of a hyper graph H is the minimum number of colours needed to colour the vertices of H such that there is no edge with its vertices all in one colour. An alternating sequence of distinct vertices and edges $v_0, e_1, v_1, ..., v_{k-1}, e_k, v_0$ is a cycle of length k in H if $v_i \in e_i \cap e_{i+1}$ for $i = 0, 1, ..., k-1$ modulo k. The length of the shortest cycle in H is the girth of H. A classic result of Erdős and Hajnal states that there exist r-uniform hypergraphs with chromatic number k and no cycle of length less than q for any given integers k and q [4]. We will use this result in the proof of our main theorem.

Theorem 1 *For any given positive integers q and k there exists a graph G_k such that $\chi_{CAP}(G) = k$, $\chi(G) = k + 1$ and G_k does not contain any cycles of length less than q.*

Proof. We prove the theorem by induction on k. For $k = 1$, we can choose K_2 to be the graph G_1. Suppose that the graph G_k has been constructed such that $\chi_{CAP}(G_k) = k$, $\chi(G_k) = k + 1$ and G_k does not contain any cycles of length less than q. Let $m_k = |V(G_k)|$ and $r = (k + 1) m_k$. By the result of Erdős and Hajnal, there exists an r-uniform hypergraph H such that $\chi(H) = k + 2$ and H does not contain any cycles of length less than q. We construct G_{k+1} by starting with $V(H)$ with no edge. For each edge $e \in E(H)$, we add $k + 1$ pairwise disjoint new copies of G_k: $G_k^{(e,1)}$, $G_k^{(e,2)}$, ..., $G_k^{(e,k+1)}$, and join their vertices to the vertices of e in $V(H)$ by a matching as demonstrated in Figure 1. The result is the graph G_{k+1} which has $|V(H)| + r|E(H)|$ vertices and $(k + 1)|E(H)||E(G_k)| + r|E(H)|$ edges. Note that there will still be no edge between any two vertices in $V(H)$.



Observation 1: $\chi(G_{k+1}) \leq k + 2$.

All the vertices in the copies of G_k can be coloured with colours $\{1, 2, ..., k+1\}$. The vertices in $V(H)$ can be coloured with colour $k+2$ since they form an independent set in G_{k+1}.

Observation 2: $\chi_{CAP}(G_{k+1}) \geq k+1$.

We show that player A has a winning strategy using colours $\{1, 2, ..., k+1\}$. For each $e \in E(H)$, player A will colour the edges having one end in $G_k^{(e,i)}$ and another end in $V(H)$ with colour i, for $i = 1, 2, ..., k+1$. For the edges between two vertices in $G_k^{(e,i)}$, player A will colour them according to the winning strategy on graph G_k with colours $\{1, ..., k+1\} \setminus \{i\}$ for $i = 1, 2, ..., k+1$. Such a strategy exists since $\chi_{CAP}(G_k) = k$.

When player B colours the vertices in $V(H)$ with colours $\{1, 2, ..., k+1\}$, the vertices corresponding to one of the edges in H will be coloured with the same colour since $\chi(H) = k+2$. Suppose that all the vertices in an edge e are coloured with colour 1. If there is a vertex in $G_k^{(e,1)}$ that is coloured with colour 1, that causes a monochromatic edge between $V(H)$ and $G_k^{(e,1)}$. On the other hand, if all the vertices in $G_k^{(e,1)}$ are coloured with colours in $\{2, 3, ..., k+1\}$, since the edges in $G_k^{(e,1)}$ are coloured according to a winning strategy of player A with colours $\{2, 3, ..., k+1\}$, there will be a monochromatic edge in $G_k^{(e,1)}$. This concludes the proof of Observation 2.

Since we have $\chi_{CAP}(G_{k+1}) \leq \chi(G_{k+1}) - 1$, combining observations 1 and 2 we would have $\chi_{CAP}(G_{k+1}) = k+1$ and $\chi(G_{k+1}) = k+2$. It remains to show that G_{k+1} does not contain a cycle of length less than q.

Let C be a cycle in G_{k+1}.

Case 1: C is contained in one copy of G_k. In this case the length of C is at least q by the inductive hypothesis.

Case 2: C is not contained in one copy of G_k. In this case, C is in the form

$$u_1 P_1 u_2 P_2 u_3 ... u_l P_l u_1$$

where $u_i \in V(H)$ and P_i is a path in a copy of G_k for $i = 1, 2, ..., l$. According to the construction of G_{k+1},that u_j and u_{j+1} are both adjacent to the ends of path P_j implies that u_j and u_{j+1} are contained in an edge in H. Then $u_1 u_2 ... u_l u_1$ is a cycle in H. Since the girth of H is at least q, we have $l \geq q$ and the length of C is at least q. ■

3 Remarks

The original proof of the result of Erdős and Hajnal was probabilistic. Constructive proofs were obtained by Lovász [8] and by Nešetřil and Rödl [9]. Therefore the graphs in the proof of Theorem 1 can be actually constructed, even though the numbers of vertices and edges are extremely large. In this paper, we did not attempt to construct the graphs with the least number of vertices. In fact, as Kostochka and Nešetřil pointed out in [7], in order for the girth of the graphs to be at least q, we only need to have the girth of the hypergraph H to be at least $\lceil \frac{q}{3} \rceil$. It would be interesting to construct (relatively) small graphs with $\chi_{CAP} = \chi - 1$ and girth at least 4 or 5. For the interested reader, there are more open problems concerning the chromatic capacity of graphs discussed in [6].

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